Trabajo de Fin de Máster en Investigación Matemática,
Curso académico 2008/2009,
Facultad de Ciencias Matemáticas,
Universidad Complutense de Madrid.

# Applications of the absolute quadratic complex and the quadric of segments in 3D reconstruction.

*Author:* Juan M. Bello Rivas

*Supervisor:* Antonio Valdés Morales

MSC2000 (Mathematics Subject Classification): 14Nxx and 51Nxx.

**Autorización de difusión:**

# Contents

# Chapter 1

# Introduction

## 1.1 Introduction in English

When a source of light illuminates a scene, rays are partially absorbed by objects in the scene and some of those rays are registered by the retina of a modern digital camera, contributing to the formation of an image. But how to make sense of those images? How can two-dimensional data be automatically converted into a representation that lets a computer 'reason' spatially about the original scene? Computer Vision is a discipline that revolves around enabling machines to make decisions based on images and, because of this, being able to reliably recover the metric three-dimensional structure of a scene from its images is an important undertaking.

This dissertation is a self-contained survey of the theory underlying certain geometric entities that help solve the problem of metric three-dimensional reconstruction. It builds upon the contributions of many researchers in the field whose work is appropriately covered and referenced in the excellent textbook by Hartley and Zisserman. Writing this document comprised the understanding of current research in Computer Vision, the coherent development of the interrelationships between the subjects, and filling the gaps (such as proofs) in the original papers. The first four chapters introduce the problem at hand following the treatment of [HZ04] while chapters five and six deal with line geometry and two recent approaches for computing metric reconstructions introduced in [RVG08] and [RV10].

I would like to thank my supervisor Antonio Valdés for his insights and willingness to help and José Ignacio Ronda from the Image Processing Group at the Universidad Politécnica de Madrid for his support.

Keywords:

- 3D reconstruction

- Projective geometry
- Absolute conic
- Absolute dual quadric
- Line geometry
- Plücker coordinates
- Absolute quadratic complex
- Quadric of segments

## 1.2 Introducción en Español

Cuando una fuente de luz ilumina una escena, los rayos son parcialmente absorbidos por los objetos de la escena y algunos de esos rayos se registran en la retina de una cámara digital moderna contribuyendo a la formación de una imagen. ¿Cómo comprender esas imágenes? ¿Cómo convertir automáticamente los datos bidimensionales en una representación que permita a un ordenador realizar 'razonamientos' espaciales acerca de la escena original? La Visión Artificial es una materia que gira alrededor de la capacitación de las máquinas para que tomen decisiones basadas en imágenes y, por esto, ser capaz de recuperar con fiabilidad la estructura tridimensional de una escena a partir de sus imágenes es una empresa importante.

Esta disertación es un estudio autocontenido de la teoría subyacente a ciertos entes geométricos que ayudan a resolver el problema de la reconstrucción métrica tridimensional. Se basa en las contribuciones de muchos investigadores en este campo cuyo trabajo está apropiadamente expuesto y citado en el excelente libro de texto de Hartley y Zisserman. Escribir este documento consistió en la comprensión de líneas de investigación actuales en Visión Artificial, en el desarrollo coherente de las interrelaciones entre las materias y en rellenar los huecos (por ejemplo, las demostraciones) de los artículos originales. Los primeros cuatro capítulos introducen el problema a resolver siguiendo el tratamiento dado en [HZ04] mientras que los capítulos cinco y seis versan sobre geometría de rectas y sobre dos enfoques recientes para la obtención de reconstrucciones métricas que se han introducido en [RVG08] y [RV10].

Me gustaría agradecer a mi tutor Antonio Valdés sus explicaciones y su voluntad de ayudarme y a José Ignacio Ronda del Grupo de Tratamiento de Imágenes de la Universidad Politécnica de Madrid por su apoyo.

Palabras clave:

- Reconstrucción 3D
- Geometría proyectiva
- Cónica del absoluto
- Dual de la cónica del absoluto
- Geometría de rectas
- Coordenadas de Plücker
- Complejo cuadrático absoluto
- Cuádrica de segmentos

# Chapter 2

# Camera models.

## 2.1 Finite projective cameras.

**Definition 1.** A *camera* is a linear mapping between $\mathbb{P}^3$ and $\mathbb{P}^2$ of the form

$$\mathtt{P} = (\mathtt{M} \,|\, \mathbf{m}) \tag{2.1}$$

where $\mathtt{M}$ is a $3 \times 3$ real matrix and $\mathbf{m}$ is a vector in $\mathbb{R}^3$.

In order to better understand the action of the camera just defined we need the following result whose proof can be found, for example, in [GvL96].

**Theorem 2.1.1.** A $n \times n$ real matrix $\mathtt{A}$ can be decomposed as

$$\mathtt{A} = \mathtt{RQ}$$

where $\mathtt{R}$ and $\mathtt{Q}$ are both $n \times n$ matrices with the former being upper triangular and the latter being orthogonal. This decomposition is known as RQ factorization.

Decomposing $\mathtt{M}$ as a product of $\mathtt{K}$ and $\mathtt{R}$ (respectively upper-triangular and orthogonal) leads us to

$$\mathtt{P} = (\mathtt{M} \,|\, \mathbf{m}) = (\mathtt{KR} \,|\, \mathbf{m}).$$

Cameras with $\mathtt{K}$ invertible are referred to as *finite projective cameras* and those with a singular $\mathtt{K}$ are called *general projective cameras*. In the former case we can write

$$\mathtt{P} = \mathtt{KR}(\mathtt{I} \,|\, -\tilde{\mathbf{C}}) = \mathtt{K}(\mathtt{R} \,|\, \mathbf{t}). \tag{2.2}$$

where $\mathbf{t} = -\mathtt{R}\tilde{\mathbf{C}}$. This is the type of cameras that we are going to study.

**Definitions 1.** The center of projection (the origin of coordinates in our case) receives the name of *camera (or optical) center*. The line passing through the camera center, orthogonal to the image plane is called the *principal line*. The point where the principal line intersects the image plane is referred to as the *principal point*. Finally, The plane parallel to the image plane passing through the camera center is called the *principal plane*.

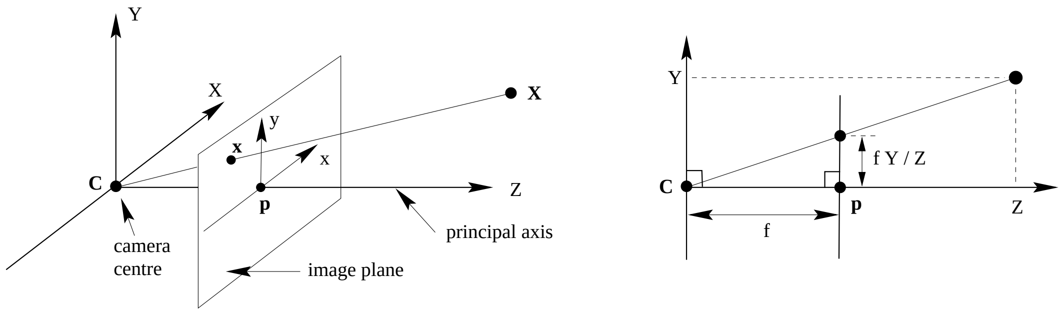


Figure 2.1: Central projection model of a pinhole camera.

Let us consider a Euclidean coordinate system for $\mathbb{R}^3$, in which the center of projection is the origin and let us call the plane $Z = f$ the *image (or focal) plane*. The reference frame just described is referred to as the *camera coordinate frame*. A point $\mathbf{X} = (X, Y, Z)^{\mathtt{T}}$ will be mapped to the point of intersection $\mathbf{x}$ between the image plane and the line joining the camera center as shown in figure 2.1.

If we decouple the *world coordinate frame* from the camera coordinate frame then both coordinate frames will be related by a rotation and a translation from the former to the latter. Letting $\tilde{\mathbf{X}}, \tilde{\mathbf{X}}_{cam} \in \mathbb{A}^3$ be (respectively) the representations of a point in the world coordinate frame and the same point in the camera coordinate frame, we can write

$$\tilde{\mathbf{X}}_{cam} = \mathtt{R}\left(\tilde{\mathbf{X}} - \tilde{\mathbf{C}}\right) \tag{2.3}$$

where $\tilde{\mathbf{C}} \in \mathbb{A}^3$ is the camera center in the world coordinate frame. And homogenizing 2.3 we can write:

$$\mathbf{X}_{cam} = \begin{pmatrix} \mathtt{R} & -\mathtt{R}\tilde{\mathbf{C}} \\ \mathbf{0} & 1 \end{pmatrix} \mathbf{X} \tag{2.4}$$

Giving

$$\mathbf{x} = \mathtt{KR}(\mathtt{I}| - \tilde{\mathbf{C}})\mathbf{X}. \tag{2.5}$$

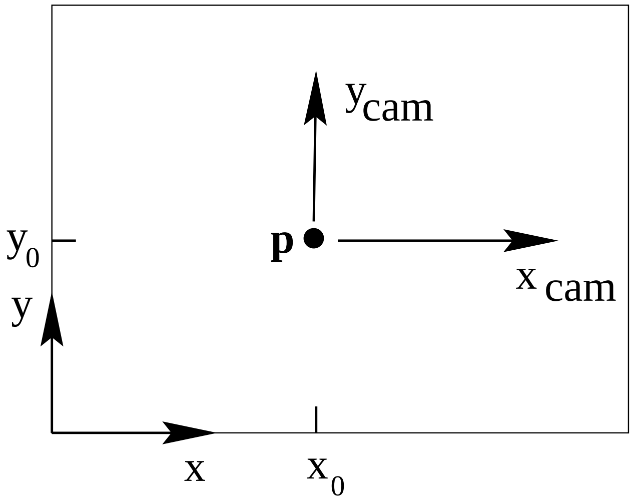


Figure 2.2: Anatomy of the image plane.

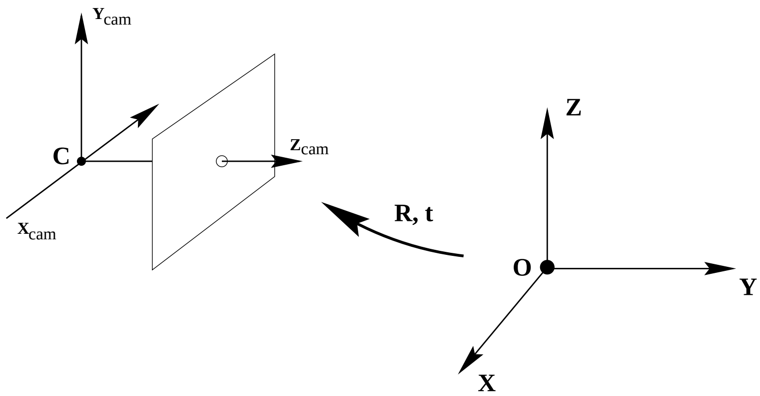


Figure 2.3: Change from world coordinate frame to camera coordinate frame.

which is consistent with our camera model 2.2.

The matrix $\mathtt{K}$ is called the *internal parameters matrix* since it doesn't depend on external (to the camera) coordinate systems while $\mathtt{R}$ and $\tilde{\mathbf{C}}$ are known as the *external parameters* for the opposite reason. The general form of $K$ is

$$\mathtt{K} = \begin{pmatrix} \alpha_x & s & x_0 \\ 0 & \alpha_y & y_0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.6}$$

Let's take a closer look at the interpretation of the entries of 2.6:

- The entries labeled $m_x$ and $m_y$ denote the number of pixels per unit distance in image coordinates in the $x$ and $y$ directions respectively. This is appropriate because a camera using a charge-coupled device (CCD) such as a modern digital camera may have non-square pixels [Wik09].

- Entries labeled $\alpha_x = fm_x$ and $\alpha_y = fm_y$ represent the focal length in each direction in terms of pixel dimensions.
- Also, $x_0 = m_x p_x$, $y_0 = m_y p_y$ are the coordinates of the principal point in terms of pixel dimensions.
- The final parameter $s$ allows for the possibility that pixels may not be rectangular either by design or due to defects in manufacturing. This extra parameter $s$ is known as *skewness* and may also appear as an artifact of numerical processing.

To summarize, let us recollect the parameters that determine a finite projective camera

- 3 degrees of freedom for the translation given by $\mathbf{t}$.
- 3 degrees of freedom for the rotation given by $\mathtt{R}$ (think of the three Euler angles).
- 2 degrees of freedom for the coordinates of the principal point.
- 1 degree of freedom for the ratio of pixel dimensions.
- 1 degree of freedom for focal length.
- 1 degree of freedom for skewness.

This makes a total of 11 degrees of freedom.

The matrix for a finite projective camera $\mathtt{P}$ has a 1-dimensional null space and we can denote by $\mathbf{C}$ the homogeneous vector generating the null space. Thus, we have

$$\mathtt{P}\mathbf{C} = \mathbf{0} \tag{2.7}$$

Now, the ray joining $\mathbf{C}$ and any other point $\mathbf{A} \in \mathbb{P}^3$ can be written as

$$\mathbf{X}(\lambda) = \lambda\mathbf{A} + (1-\lambda)\mathbf{C} \tag{2.8}$$

and, consequently, the image of any point in this ray is

$$\mathbf{x} = \mathtt{P}\mathbf{X}(\lambda) = \lambda\mathtt{P}\mathbf{A} + (1-\lambda)\mathtt{P}\mathbf{C} = \lambda\mathtt{P}\mathbf{A} \tag{2.9}$$

By 2.7, this means that points on the ray collapse to the same point $\lambda\mathtt{P}\mathbf{A}$ in the image plane which is equivalent to stating that the ray $\mathbf{X}(\lambda)$ is a ray through the camera center.

A more in-depth treatment of camera anatomy can be found in [HZ04].

## 2.2 Action of a projective camera on points

### 2.2.1 Forward projection

While developing the model of a projective camera we've seen how it projects real points to the image plane. A treatment of how points at infinity are dealt with is now in order.

From the model we have developed it is clear that points $\mathbf{X}$ from 3D space are mapped by $\mathtt{P}$ to points $\mathbf{x}$ in the image plane. Now let's consider a point $\mathbf{D} = (\mathbf{d}^{\mathsf{T}}, 0)^{\mathsf{T}}$ at the plane at infinity $\pi_\infty$ in a Euclidean reference frame (those points represent directions or vanishing points). We can see that if we write $\mathtt{P}$ as the juxtaposition of a $3 \times 3$ matrix $\mathtt{M}$ and a vector $\mathbf{p}$ then,

$$\mathbf{x} = \mathtt{P}\mathbf{D} = (\mathtt{M}|\mathbf{p})\mathbf{D} = \mathtt{M}\mathbf{d} \tag{2.10}$$

which means that the mapping of points at infinity to image points is exclusively determined by $\mathtt{M}$.

### 2.2.2 Back-projection of points to rays

We want to map an image point $\mathbf{x}$ to a ray containing points which will forward-project again to $\mathbf{x}$. To do this, we use two pieces of information:

1. the camera center $\mathbf{C}$
2. the point $\mathtt{P}^+\mathbf{x}$ where $\mathtt{P}^+ = \mathtt{P}^{\mathsf{T}}\left(\mathtt{P}\mathtt{P}^{\mathsf{T}}\right)^{-1}$ is the pseudoinverse matrix of $\mathtt{P}$ (see [HZ04]).

Therefore, the line we are looking for is:

$$\mathbf{X}(\lambda) = \lambda\mathtt{P}^+\mathbf{x} + (1-\lambda)\mathbf{C} = \mathtt{P}^+x + \lambda\mathbf{C} \tag{2.11}$$

## 2.3 Action of a projective camera on lines

### 2.3.1 Forward projection

Let $\mathbf{A}, \mathbf{B}$ be points in $\mathbb{P}^3$. The line joining those points admits a parametric representation of the form

$$\mathbf{X}(\mu) = \mathbf{A} + \mu\mathbf{B}$$

where $\mu$ is an inhomogeneous projective parameter that can be $\infty$. Points $\mathbf{X}(\mu)$ will be projected to points of the form

$$\mathbf{x}(\mu) = \mathtt{P}\mathbf{X}(\mu) = \mathtt{P}\left(\mathbf{A} + \mu\mathbf{B}\right) = \mathtt{P}\mathbf{A} + \mu\mathtt{P}\mathbf{B} = \mathbf{a} + \mu\mathbf{b}$$

which implies that lines in $\mathbb{P}^3$ are projected to lines contained in the image plane.

### 2.3.2 Back-projection of lines to planes

An imaged line $\mathbf{l}$ back-projects to the plane $\mathtt{P}^\mathsf{T}\mathbf{l}$ since a point $\mathbf{X}$ is projected as a point in the line $\mathbf{l}$ if and only if $(\mathtt{P}\mathbf{X})^\mathsf{T}\mathbf{l} = \mathbf{X}^\mathsf{T}(\mathtt{P}^\mathsf{T}\mathbf{l}) = 0$.

# Chapter 3

# The absolute conic and its dual quadric

## 3.1 The absolute conic.

In a Euclidean reference frame, the absolute conic $\Omega_\infty$ is the set of points of the form $\mathbf{x} = (x_0, x_1, x_2, x_3)^\mathtt{T}$ that simultaneously satisfy the following equations

$$\begin{cases} x_0^2 + x_1^2 + x_2^2 & = 0 \\ & x_3 = 0 \end{cases} \tag{3.1}$$

This means that $\Omega_\infty$ is a conic of points at infinity (directions or, equivalently, points having $x_3 = 0$ in a Euclidean reference frame) that can be represented in matrix notation as follows

$$(x_0, x_1, x_2)^\mathtt{T} \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = 0 \tag{3.2}$$

Since $\Omega_\infty \subset \boldsymbol{\pi}_\infty$, the absolute conic is not the projective completion of any affine conic.

### 3.1.1 Properties

**Proposition 3.1.1.** The absolute conic $\Omega_\infty$ is fixed under an homography $\mathtt{H}$ if and only if $\mathtt{H}$ is a similarity transformation.

Some observations are in order:

1. In general, each point in the absolute conic doesn't remain fixed by similarities but the conic does.

2. All spheres intersect the plane at infinity in the absolute conic. To see this it is enough to homogenize the equation corresponding to a sphere and check that this is indeed the case.

3. Circles intersect the absolute conic in two points. Namely, the cyclic points at infinity which are represented by $(1, i, 0)^\mathtt{T}$ and $(1, -i, 0)^\mathtt{T}$ in a Euclidean reference frame.

### 3.1.2 Orthogonality as a projective invariant

By 3.2, we can regard two orthogonal directions $\mathbf{d}_1$ and $\mathbf{d}_2$ as being conjugate points with respect to $\Omega_\infty$.

In $\mathbb{P}^3$, if $\mathtt{C}$ is the matrix representation of a quadric (a conic being in this case a degenerate case of quadric), the coordinates of the polar plane $\boldsymbol{\pi}$ to the point $\mathbf{x}$ with respect to $\mathtt{C}$ are given by $\boldsymbol{\pi} = \mathtt{C}\mathbf{x}$.

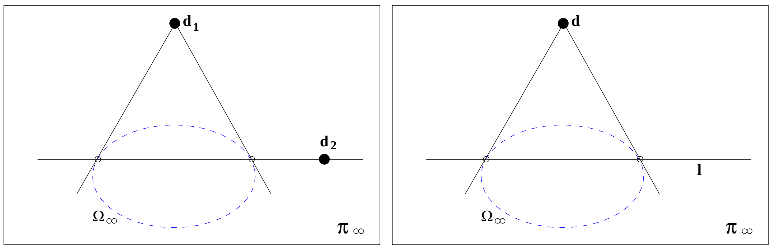


Figure 3.1: Orthogonality as conjugacy with respect to the absolute conic.

In our case, $\mathbf{d}_1$ and $\mathbf{d}_2$ are points at the plane at infinity that satisfy $\mathbf{d}_1^\mathtt{T}\Omega_\infty\mathbf{d}_2 = 0$ and, in light of what we have just discussed, $\mathbf{d}_1$ lies in the intersection the polar line to $\mathbf{d}_2$ with respect to $\Omega_\infty$. This amounts to stating that orthogonality can be interpreted as a projective invariant if the plane at infinity and the absolute conic are known.

### 3.1.3 Camera calibration and the absolute conic

A camera $\mathtt{P}$ in a camera Euclidean reference frame projects a point $\tilde{\mathbf{X}} \in \mathbb{A}^3$ to $\mathbf{x} = \mathtt{P}(\tilde{\mathbf{X}}, 1)^\mathtt{T} = \mathtt{K}(\mathtt{I} \,|\, \mathbf{0})(\tilde{\mathbf{X}}, 1)^\mathtt{T} = \mathtt{K}\tilde{\mathbf{X}}$. If $\tilde{\mathbf{X}}$ is a point along the ray joining the camera center with $\mathbf{x}$ then it will be of the form $\tilde{\mathbf{X}} = \lambda\mathbf{d}$ where $\mathbf{d}$ is the ray's direction. Its projection will be

$$\mathbf{x} = \mathtt{K}(\mathtt{I} \,|\, \mathbf{0})(\lambda\mathbf{d}, 1)^\mathtt{T} = \lambda\mathtt{K}\mathbf{d}$$

Noting that we're working with homogeneous coordinates, this leads us to the following expression,

$$\mathbf{d} = \mathtt{K}^{-1}\mathbf{x} \tag{3.3}$$

So it turns out that the camera calibration matrix $\mathtt{K}$ lets us recover the direction of the ray joining the camera center with a given image point $\mathbf{x}$. Let's denote by $\mathbf{d}_1$ and $\mathbf{d}_2$ the respective directions of the rays joining the camera center with the image points $\mathbf{x}_1$ and $\mathbf{x}_2$. By substituting 3.3 in 3.1.4, we see that $\mathtt{K}$ can also serve as a measurement device for the angle $\theta$ between rays with directions $\mathbf{d}_1$ and $\mathbf{d}_2$

$$\cos\theta = \frac{\mathbf{d}_1^{\mathsf{T}}\mathbf{d}_2}{\sqrt{(\mathbf{d}_1^{\mathsf{T}}\mathbf{d}_1)(\mathbf{d}_2^{\mathsf{T}}\mathbf{d}_2)}} = \frac{\left(\mathtt{K}^{-1}\mathbf{x}_1\right)^{\mathsf{T}}\left(\mathtt{K}^{-1}\mathbf{x}_2\right)}{\sqrt{\left((\mathtt{K}^{-1}\mathbf{x}_1)^{\mathsf{T}}(\mathtt{K}^{-1}\mathbf{x}_1)\right)\left((\mathtt{K}^{-1}\mathbf{x}_2)^{\mathsf{T}}(\mathtt{K}^{-1}\mathbf{x}_2)\right)}} \tag{3.4}$$

$$= \frac{\mathbf{x}_1^{\mathsf{T}}\left(\mathtt{K}^{-\mathsf{T}}\mathtt{K}^{-1}\right)\mathbf{x}_2}{\sqrt{\left(\mathbf{x}_1^{\mathsf{T}}(\mathtt{K}^{-\mathsf{T}}\mathtt{K}^{-1})\mathbf{x}_1\right)\left(\mathbf{x}_2^{\mathsf{T}}(\mathtt{K}^{-\mathsf{T}}\mathtt{K}^{-1})\mathbf{x}_2\right)}} \tag{3.5}$$

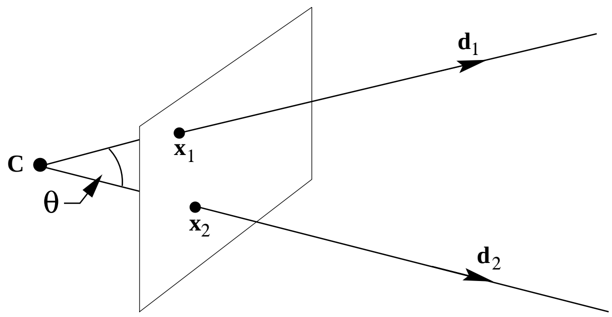


Figure 3.2: The camera calibration matrix allows us to measure angles between rays

### 3.1.4 Euclidean reconstruction from the absolute conic

The reason for the absolute conic's usefulness is its ability to recover metric properties in $\mathbb{P}^3$. After the absolute conic and the plane at infinity have been located in $\mathbb{P}^3$, we are able to measure angles and relative lengths by using 3.2 to extend to arbitrary projective coordinate systems the cosine formula

$$\cos\theta = \frac{\mathbf{d}_1^{\mathsf{T}}\mathbf{d}_2}{\sqrt{(\mathbf{d}_1^{\mathsf{T}}\mathbf{d}_1)\,(\mathbf{d}_2^{\mathsf{T}}\mathbf{d}_2)}}$$

(where $\mathbf{d}_1$ and $\mathbf{d}_2$ are directions, that is, points contained in $\pi_\infty$) thus obtaining

$$\cos\theta = \frac{\mathbf{d}_1^{\mathsf{T}}\Omega_\infty\mathbf{d}_2}{\sqrt{(\mathbf{d}_1^{\mathsf{T}}\Omega_\infty\mathbf{d}_1)\,(\mathbf{d}_2^{\mathsf{T}}\Omega_\infty\mathbf{d}_2)}}. \tag{3.6}$$

Note that we have abused notation to denote by $\mathbf{d}_i$ both a vector in 3-space and a direction in $\mathbb{P}^3$.

## 3.2 The absolute dual quadric.

We can conceive the absolute conic $\Omega_\infty$ as a limit of non-degenerate quadrics $\mathtt{Q}_k$ having matrix representations of the form $\mathrm{diag}(1,1,1,k)$ with $k \in \mathbb{R}$,

$$\mathtt{Q}_\infty = \lim_{k\to\infty} \mathrm{diag}(1,1,1,k)$$

This makes sense because as $k$ approaches infinity, the quadrics $\mathtt{Q}_k$ get closer to $\Omega_\infty$ (which is contained in $\boldsymbol{\pi}_\infty$). This is readily seen from the fact that points $\mathbf{x} = (x,y,z,w)^\mathtt{T}$ in the quadric $\mathtt{Q}_k$ satisfy

$$x_0^2 + x_1^2 + x_2^2 + kx_3^2 = 0 \tag{3.7}$$

Thus, both $x_3$ and $x_0^2 + x_1^2 + x_2^2 = 0$ must approach zero as $k$ increases. Once we have obtained the matrix representation for $\Omega_\infty$, computing the representation for the dual quadric is straight-forward

$$\mathtt{Q}^*_\infty = \mathtt{Q}^{-1}_\infty = \lim_{k\to\infty} \mathrm{diag}(1,1,1,1/k) = \mathrm{diag}(1,1,1,0) \tag{3.8}$$

We will refer to this geometric entity as the absolute dual quadric from now on. Since the absolute conic is a (degenerate) point quadric, the absolute dual quadric is a plane quadric and its planes are tangent to $\Omega_\infty$.

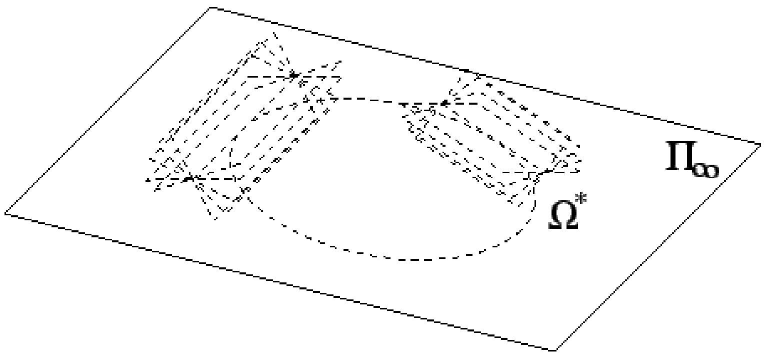


Figure 3.3: Visualization of the absolute dual quadric

### 3.2.1 Properties

**Proposition 3.2.1.** The absolute dual quadric is fixed by an homography $\mathtt{H}$ if and only if $\mathtt{H}$ is a similarity transform.

*Proof.* Let

$$\mathtt{H} = \begin{pmatrix} \mathtt{A} & \mathbf{b} \\ \mathbf{c}^{\mathtt{T}} & d \end{pmatrix}$$

be an homography in $\mathbb{P}^3$. We have that

$$\mathtt{H}\,(\mathtt{Q}^*_\infty)^{\mathrm{euc}}\,\mathtt{H}^{\mathtt{T}} \sim \begin{pmatrix} \mathtt{AA}^{\mathtt{T}} & \mathtt{A}\mathbf{c} \\ (\mathtt{A}\mathbf{c})^{\mathtt{T}} & 0 \end{pmatrix} \sim (\mathtt{Q}^*_\infty)^{\mathrm{euc}}$$

is satisfied if and only if $\mathtt{A}\mathbf{c} = \mathbf{0}$ and $\mathtt{AA}^{\mathtt{T}} \sim \mathtt{I}_3$ and this, in turn, is equivalent to $\mathtt{A} = s\mathtt{R}$ with $s \in \mathbb{R} \setminus \{0\}$ and $\mathtt{R}$ orthogonal. □

**Proposition 3.2.2.** The null space of the matrix $\mathtt{Q}^*_\infty$ is 1-dimensional and any of its generators gives the homogeneous coordinates of $\boldsymbol{\pi}_\infty$.

*Proof.* The rank of a matrix is a projective invariant and the conclusion is immediate from the expression of $\mathtt{Q}^*_\infty$ in Euclidean coordinates. □

**Proposition 3.2.3.** Let $\pi_1$ and $\pi_2$ be two planes in $\mathbb{P}^3$. The angle $\theta$ between $\pi_1$ and $\pi_2$ is given by the expression

$$\cos\theta = \frac{\pi_1^{\mathtt{T}}\mathtt{Q}^*_\infty\pi_2}{\sqrt{(\pi_1^{\mathtt{T}}\mathtt{Q}^*_\infty\pi_1)\,(\pi_2^{\mathtt{T}}\mathtt{Q}^*_\infty\pi_2)}}$$

*Proof.* This relation is projectively invariant so it is enough to prove it in Euclidean coordinates, which is trivial. □

### 3.2.2 Euclidean reconstruction from the DAQ.

The absolute dual quadric provides us with a practical method for endowing $\mathbb{P}^3$ with the ability to measure angles and ratios of lengths. Once the matrix representation of $\mathtt{Q}^*_\infty$ is estimated by some means (see [HZ04] for a detailed account of these matters), then it can be factored using the eigenvalue decomposition as

$$\mathtt{Q}^*_\infty = \mathtt{H}\tilde{\mathtt{I}}\mathtt{H}^{\mathtt{T}}$$

where $\tilde{\mathtt{I}}$ is a $4\times 4$ matrix of the form $\tilde{\mathtt{I}} = \mathrm{diag}(1,1,1,0)$. Noting that $\tilde{\mathtt{I}}$ is the matrix representation of $\mathtt{Q}^*_\infty$ in the Euclidean reference frame, it follows that $\mathtt{H}^{-1}$ is an homography which transforms *points* from the reference frame in which $\mathtt{Q}^*_\infty$ is expressed to the Euclidean reference frame.

The absolute dual quadric was introduced in the autocalibration literature by Bill Triggs in 1997 in his paper [Tri97]. A comprehensive treatment of the absolute conic and its dual quadric is provided in [SK79].

# Chapter 4

# Stratified scene reconstruction.

We will now sketch the steps of the usual approach employed in the obtainment of a 3D reconstruction. For simplicity's sake we will restrict ourselves to considering only two images of the same scene obtained from different view points.

## 4.1 Projective reconstruction.

Figure 4.1: Two images of a 3D scene.

The knowledge of sufficiently many correspondences $\{\mathbf{x}_i \leftrightarrow \mathbf{x}'_i\}$ between points in the images of a scene and the *Projective reconstruction theorem* (see [HZ04], chapter 10) make it possible to obtain a camera pair $(\mathtt{P}, \mathtt{P}')$ together with a set of points $\{\mathbf{X}_i \in \mathbb{P}^3\}$ which project to $\{\mathbf{x}_i \in \mathbb{P}^2\}$ and $\{\mathbf{x'}_i \in \mathbb{P}^2\}$ respectively under $\mathtt{P}$ and $\mathtt{P}'$ and are unique up to an homography transformation. A tuple $(\mathtt{P}, \mathtt{P}', \{\mathbf{X}_i\})$ obtained in this manner is usually referred to as a *projective calibration* of the original scene.

A projective calibration enables us to measure projective invariants of the original scene such as

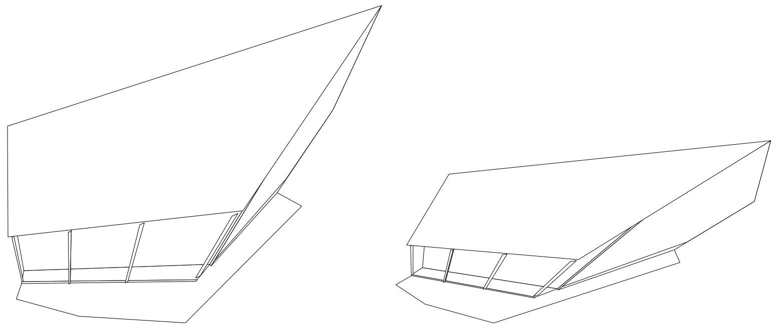

Figure 4.2: Two views of a projective reconstruction obtained from 4.1.

- cross ratio of points and planes
- intersection of planes and lines
- tangency of planes and lines

## 4.2 Affine upgrading.

If we want to measure certain affine invariants like

- parallelism of lines and planes
- volume ratios
- centers of masses

we need to locate the plane at infinity in the projective calibration. Some pieces of information that let us to achieve this goal are:

**Translational motion** Knowing that the two cameras are related by a purely translational motion lets us isolate points at infinity because they are placed at the same coordinates in both images.

**Parallel lines** Three distinct pairs of parallel lines give us 3 points in $\boldsymbol{\pi}_\infty$, which is enough to determine it.

**Distance ratios on a line** Vanishing points may be computed via the cross ratio from the knowledge of the ratios of lengths of points in a line in the original scene and their imaged points.

If $\boldsymbol{\pi} = (\pi_0, \pi_1, \pi_2, \pi_3)^{\mathtt{T}}$ and $\boldsymbol{\pi}_\infty = (0,0,0,1)^{\mathtt{T}}$ are respectively the coordinate vectors corresponding to the plane at infinity in the projective reference frame for $(\mathtt{P}, \mathtt{P}', \{\mathbf{X}_i\})$ and the plane at infinity in a Euclidean reference frame then a *point* homography $\mathtt{H}$ that transforms $\boldsymbol{\pi}$ to $\boldsymbol{\pi}_\infty$ is

$$\mathtt{H} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \pi_0 & \pi_1 & \pi_2 & \pi_3 \end{pmatrix}$$

A tuple $(\mathtt{P}, \mathtt{P}', \{\mathbf{X}_i\})$ transformed by this homography is called an *affine reconstruction.*

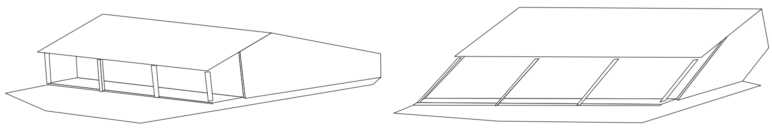

Figure 4.3: Two views of an affine reconstruction.

## 4.3 Euclidean upgrading.

If we have identified by some means the absolute conic in a projective reconstruction then we can proceed as in 3.2.2 in order to obtain a rectifying homography directly.

The image of the absolute conic, denoted by $\omega$, is a conic obtained by mapping the absolute conic $\Omega_\infty$ to the image plane by a camera $\mathtt{P}$. If the only information at our disposal is an affine reconstruction and $\omega$, we can obtain an homography (see [HZ04]) which transforms the reconstruction into another which is related to the true reconstruction by a similarity.

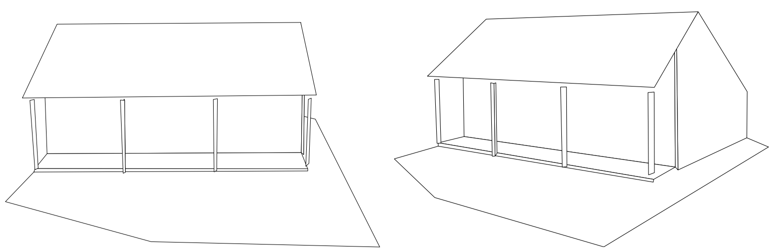

Figure 4.4: Two views of a Euclidean reconstruction.

In order to recover $\omega$ in an affine reconstruction one might, for example, exploit constraints imposed by pairs of vanishing points from orthogonal scene lines (they are conjugate with respect to the image of the absolute conic) or knowledge of a vanishing line and a vanishing point corresponding to an imaged plane and the direction of an orthogonal line (which are in polar-pole relationship with respect to $\omega$).

## 4.4 Direct reconstruction using ground truth.

A reference frame in $\mathbb{P}^3$ is given by five points so if we know the correspondence between five points in the projective reconstruction and their counterparts in a Euclidean reference frame then we can compute a rectifying homography.

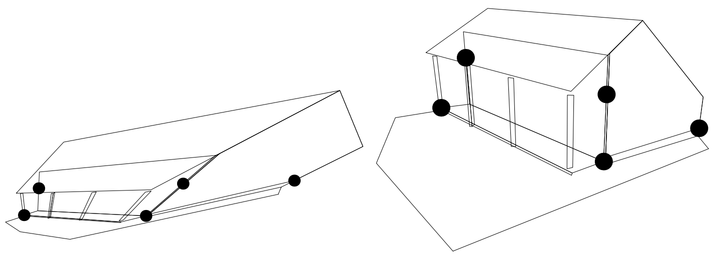

Figure 4.5: Euclidean reconstruction using ground truth.

# Chapter 5

# The absolute quadratic complex

## 5.1 Introduction

The principle of duality makes it easy to work with points and planes in $\mathbb{P}^3$ but computing with lines is not so straight-forward. In this chapter, we will develop some tools used for line geometry such as Plücker matrices and Plücker coordinates and we will explore how these tools give rise to a geometric entity equivalent to the absolute conic which appeared first in [PMP$^+$05]. A general introduction to line geometry and line complexes can be found in [SK79], for a comprehensive treatment of line geometry including many industrial applications, see [PW01]. Also, Grassman-Cayley algebra and Clifford algebra provide adequate generalizations of this subject.

## 5.2 Plücker matrices

Let $\mathbf{u} = (u_1, u_2, u_3, u_4)^\mathtt{T}$ and $\mathbf{v} = (v_1, v_2, v_3, v_4)^\mathtt{T}$ be two vectors in $\mathbb{C}^4$. Let's define the matrix

$$\mathtt{M}(\mathbf{u}, \mathbf{v}) = \mathbf{u}\mathbf{v}^\mathtt{T} - \mathbf{v}\mathbf{u}^\mathtt{T} = \begin{pmatrix} 0 & m_{12} & m_{13} & m_{14} \\ -m_{12} & 0 & m_{23} & m_{24} \\ -m_{13} & -m_{23} & 0 & m_{34} \\ -m_{14} & -m_{24} & -m_{34} & 0 \end{pmatrix}$$

where

$$m_{ij} = u_i v_j - u_j v_i = \begin{vmatrix} u_i & v_i \\ u_j & v_j \end{vmatrix}$$

This antisymmetric matrix is of rank two if and only if $\mathbf{u}$ and $\mathbf{v}$ are independent and of rank zero otherwise because its columns are linear combinations of $\mathbf{u}$ and $\mathbf{v}$.

**Definition 2.** The $\mathtt{L}$-matrix of a line joining two points $\mathbf{p}$ and $\mathbf{q}$ in $\mathbb{P}^3$ is the matrix $\mathtt{L} = \mathtt{M}(\mathbf{p}, \mathbf{q})$.

By the properties of the determinant 5.2, two $\mathtt{L}$-matrices $\mathtt{M}(\mathbf{p}, \mathbf{q})$ and $\mathtt{M}(\mathbf{p'}, \mathbf{q'})$ corresponding to the same line are equal up to a non-zero scale factor.

**Definition 3.** Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)^{\mathtt{T}}$ and $\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3, \beta_4)^{\mathtt{T}}$ be two planes in $\mathbb{P}^3$. We define the $\mathtt{L}^*$-matrix of the line determined by the intersection of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ as $\mathtt{L}^* = \mathtt{M}(\boldsymbol{\alpha}, \boldsymbol{\beta})$.

### 5.2.1 Basic relations

**Lemma 5.2.1.** A non-zero $4 \times 4$ singular antisymmetric matrix $\mathtt{A}$ is of rank 2.

*Proof.* Inspecting the eigenvalues of

$$\mathtt{A} = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix},$$

which are given by the roots of the characteristic polynomial

$$\lambda^4 + \left( \sum_{i=1}^{4} \sum_{j=i+1}^{4} a_{ij}^2 \right) \lambda^2 = 0,$$

it is immediate to see that the rank of $\mathtt{A}$ is 2. □

**Lemma 5.2.2.** A non-zero $4 \times 4$ singular antisymmetric matrix $\mathtt{A}$ is determined by its null space up to a non-zero scale factor.

*Proof.* Let $\mathbf{u_1}, \mathbf{u_2}$ be two vectors spanning the null space of $\mathtt{A}$. We can perform a change of coordinates such that $\mathbf{u_i} \mapsto \mathbf{e_i}$ with $i \in \{1, 2\}$ where the $\mathbf{e_i}$ are the first two vectors in the canonical basis for $\mathbb{C}^4$. The matrix $\mathtt{A}$ is transformed accordingly into an antisymmetric matrix $\mathtt{A}'$ such that $\mathtt{A}'\mathbf{e_1} = \mathtt{A}'\mathbf{e_2} = \mathbf{0}$, which means that the only non-zero entries in $\mathtt{A}'$ (and consequently in $\mathtt{A}$) are $a'_{34} = -a'_{43}$. Thus, $\mathtt{A}$ is defined up to a non-zero scale factor. □

**Theorem 5.2.1.** A non-zero antisymmetric matrix $\mathtt{M} = (m_{ij})_{i,j=1}^{4} \in \mathbb{C}^{4\times 4}$ represents a line if and only if

$$m_{12}m_{34} + m_{13}m_{42} + m_{23}m_{14} = 0 \tag{5.1}$$

*Proof.* If $\mathtt{M}$ represents a line it must have a non-zero kernel and then

$$\det \mathtt{M} = (m_{12}m_{34} + m_{13}m_{42} + m_{23}m_{14})^2 = 0, \tag{5.2}$$

which implies that

$$m_{12}m_{34} + m_{13}m_{42} + m_{23}m_{14} = 0. \tag{5.3}$$

And Lemma 5.2.2 ensures $\mathtt{M}$ is unique up to a non-zero scale factor.

Conversely, if the entries of $\mathtt{M}$ satisfy 5.1 then $\det \mathtt{M} = 0$ which means it is rank-deficient but, by Lemma 5.2.1, we know that $\dim \ker \mathtt{M} = 2$ (otherwise it would be the zero matrix) and, consequently, $\mathtt{M}$ represents a line. □

*Remark* 5.2.1. The kernel of $\mathtt{L}$ is the pencil of planes having the corresponding line as its axis which, together with Lemma 5.2.1, means that this matrix completely determines the line.

**Proposition 5.2.1.** Let $\mathtt{L}$ be the $\mathtt{L}$-matrix of a certain line $\boldsymbol{\ell}$ and let $\boldsymbol{\alpha}$ be a plane not containing $\boldsymbol{\ell}$. Then $\mathbf{x} = \mathtt{L}\boldsymbol{\alpha}$ is the point of intersection between the line $\boldsymbol{\ell}$ and the plane $\boldsymbol{\alpha}$.

*Proof.* If $\boldsymbol{\ell}$ is the line joining two points $\mathbf{p}, \mathbf{q} \in \mathbb{P}^3$ then $\mathtt{L} = \mathtt{M}(\mathbf{p}, \mathbf{q}) = \mathbf{p}\mathbf{q}^{\mathtt{T}} - \mathbf{q}\mathbf{p}^{\mathtt{T}}$ and $\mathbf{x} = \mathtt{L}\boldsymbol{\alpha}$ is a linear combination of $\mathbf{p}$ and $\mathbf{q}$, which implies that $\mathbf{x} \in \boldsymbol{\ell}$.

Now, $\mathbf{x}^{\mathtt{T}}\boldsymbol{\alpha} = 0$ because the terms at the left and right hand sides of the equation

$$\mathbf{x}^{\mathtt{T}}\boldsymbol{\alpha} = (\mathtt{L}\boldsymbol{\alpha})^{\mathtt{T}}\boldsymbol{\alpha} = -\boldsymbol{\alpha}^{\mathtt{T}}(\mathtt{L}\boldsymbol{\alpha}) = -\boldsymbol{\alpha}^{\mathtt{T}}\mathbf{x}$$

are scalars. □

**Definition 4.** The *Plücker matrices* of a line are its associated $\mathtt{L}$-matrix and $\mathtt{L}^*$-matrix .

To see how the different types of Plücker matrices are related we are going to introduce a new antisymmetric matrix that is defined as the unique matrix that satisfies

$$\det(\mathbf{x}, \mathbf{u}, \mathbf{v}, \mathbf{y}) = \mathbf{x}^{\mathtt{T}}\mathtt{M}^*(\mathbf{u}, \mathbf{v})\mathbf{y} \tag{5.4}$$

for $\mathbf{x}, \mathbf{u}, \mathbf{v}, \mathbf{y} \in \mathbb{C}^4$. It has the explicit expression

$$\mathtt{M}^*(\mathbf{u}, \mathbf{v})\,\mathbf{x} = \begin{pmatrix} 0 & m_{34} & m_{42} & m_{23} \\ -m_{34} & 0 & m_{14} & m_{31} \\ -m_{42} & -m_{14} & 0 & m_{12} \\ -m_{23} & -m_{31} & -m_{12} & 0 \end{pmatrix} \mathbf{x} = \mathbf{0} \tag{5.5}$$

with $m_{ij}$ as in 5.2.

*Remark* 5.2.2. From relation 5.4 it is clear that the kernel of $\mathtt{M}^*(\mathbf{p},\mathbf{q})$ is the range of points in the line through $\mathbf{p}$ and $\mathbf{q}$ and, by Lemma 5.2.2, this means that $\mathtt{M}^*(\mathbf{p},\mathbf{q}) \sim \mathtt{M}(\boldsymbol{\alpha},\boldsymbol{\beta})$. Consequently, $\mathtt{M}^*(\mathbf{p},\mathbf{q})$ is an $\mathtt{L}^*$-matrix of the line defined by planes $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. In a similar fashion, $\mathtt{M}^*(\boldsymbol{\alpha},\boldsymbol{\beta})$ is an $\mathtt{L}$-matrix of the line defined by points $\mathbf{p}$ and $\mathbf{q}$. Hence, $\mathtt{M}^*(\boldsymbol{\alpha},\boldsymbol{\beta}) \sim \mathtt{M}(\mathbf{p},\mathbf{q})$

### 5.2.2 Incidence of lines

**Proposition 5.2.2.** Let $\boldsymbol{\ell}_1$ and $\boldsymbol{\ell}_2$ be two lines and consider an $\mathtt{L}$-matrix $\mathtt{L_1}$ and an $\mathtt{L}^*$-matrix $\mathtt{L}_2^*$ corresponding to $\boldsymbol{\ell}_1$ and $\boldsymbol{\ell}_2$ respectively.

*Proof.* If $\boldsymbol{\ell}_1$ and $\boldsymbol{\ell}_2$ are given respectively by points $\mathbf{p_1},\mathbf{q_1}$ and $\mathbf{p_2},\mathbf{q_2}$ then they intersect if and only if the four points $\mathbf{p_1},\mathbf{q_1},\mathbf{p_2},\mathbf{q_2}$ are coplanar. This can be expressed as

$$\det(\mathbf{p_1},\mathbf{q_1},\mathbf{p_2},\mathbf{q_2}) = 0$$

and we can write it in terms of $\mathtt{L_1}$ and $\mathtt{L}_2^*$ as follows:

$$\begin{aligned}
0 &= \det(\mathbf{p_1},\mathbf{q_1},\mathbf{p_2},\mathbf{q_2}) \\
&= \frac{1}{2}\left(\det(\mathbf{q_1},\mathbf{p_2},\mathbf{q_2},\mathbf{p_1}) - \det(\mathbf{p_1},\mathbf{p_2},\mathbf{q_2},\mathbf{q_1})\right) \\
&= \frac{1}{2}\left(\mathbf{q_1}^{\mathtt{T}}\mathtt{M}^*(\mathbf{p_2},\mathbf{q_2})\mathbf{p_1} - \mathbf{p_1}^{\mathtt{T}}\mathtt{M}^*(\mathbf{p_2},\mathbf{q_2})\mathbf{q_1}\right) \\
&= \frac{1}{2}\operatorname{trace}\left(\mathbf{q_1}^{\mathtt{T}}\mathtt{M}^*(\mathbf{p_2},\mathbf{q_2})\mathbf{p_1} - \mathbf{p_1}^{\mathtt{T}}\mathtt{M}^*(\mathbf{p_2},\mathbf{q_2})\mathbf{q_1}\right) \\
&= \frac{1}{2}\operatorname{trace}\left(\mathbf{q_1}^{\mathtt{T}}\mathtt{M}^*(\mathbf{p_2},\mathbf{q_2})\mathbf{p_1}\right) - \frac{1}{2}\operatorname{trace}\left(\mathbf{p_1}^{\mathtt{T}}\mathtt{M}^*(\mathbf{p_2},\mathbf{q_2})\mathbf{q_1}\right) \\
&= \frac{1}{2}\operatorname{trace}\left(\mathbf{p_1}\mathbf{q_1}^{\mathtt{T}}\mathtt{M}^*(\mathbf{p_2},\mathbf{q_2})\right) - \frac{1}{2}\operatorname{trace}\left(\mathbf{q_1}\mathbf{p_1}^{\mathtt{T}}\mathtt{M}^*(\mathbf{p_2},\mathbf{q_2})\right) \\
&= \frac{1}{2}\operatorname{trace}\left(\mathbf{p_1}\mathbf{q_1}^{\mathtt{T}}\mathtt{M}^*(\mathbf{p_2},\mathbf{q_2}) - \mathbf{q_1}\mathbf{p_1}^{\mathtt{T}}\mathtt{M}^*(\mathbf{p_2},\mathbf{q_2})\right) \\
&= \frac{1}{2}\operatorname{trace}\left(\left(\mathbf{p_1}\mathbf{q_1}^{\mathtt{T}} - \mathbf{q_1}\mathbf{p_1}^{\mathtt{T}}\right)\mathtt{M}^*(\mathbf{p_2},\mathbf{q_2})\right) \\
&= \frac{1}{2}\operatorname{trace}\left(\mathtt{M}(\mathbf{p_1},\mathbf{q_1})\mathtt{M}^*(\mathbf{p_2},\mathbf{q_2})\right) = \frac{1}{2}\operatorname{trace}\left(\mathtt{L_1}\mathtt{L}_2^*\right)
\end{aligned}$$

□

**Proposition 5.2.3.** Let $\boldsymbol{\ell}_1$ and $\boldsymbol{\ell}_2$ be two distinct, intersecting lines with associated Plücker matrices $\mathtt{L_1}$ and $\mathtt{L}_2^*$. Then any non-zero column of the product $\mathtt{L_1}\mathtt{L}_2^*$ represents the intersection point.

*Proof.* We can suppose, without loss of generality, that $\boldsymbol{\ell}_2$ is determined by two planes $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ such that $\boldsymbol{\ell}_1$ is contained in $\boldsymbol{\alpha}$. By Proposition 5.2.1, the columns of the product

$$\begin{aligned}\mathtt{L}_1\mathtt{L}_2^* &= \mathtt{L}_1\mathtt{M}(\boldsymbol{\alpha},\boldsymbol{\beta}) = \mathtt{L}_1\left(\boldsymbol{\alpha}\boldsymbol{\beta}^{\mathtt{T}} + \boldsymbol{\beta}\boldsymbol{\alpha}^{\mathtt{T}}\right)\\ &= (\mathtt{L}_1\boldsymbol{\alpha})\boldsymbol{\beta}^{\mathtt{T}} + (\mathtt{L}_1\boldsymbol{\beta})\boldsymbol{\alpha}^{\mathtt{T}} = (\mathtt{L}_1\boldsymbol{\beta})\boldsymbol{\alpha}^{\mathtt{T}}\end{aligned}$$

are multiples of the point of intersection between $\boldsymbol{\ell_1}$ and $\boldsymbol{\beta}$. □

*Remark* 5.2.3. As an immediate consequence of the previous proposition, we can state that three lines are concurrent if and only if $\mathtt{L}_1^*\mathtt{L}_2\mathtt{L}_3^* = \mathtt{0}$. And, dually, three lines are coplanar if and only if $\mathtt{L}_1\mathtt{L}_2^*\mathtt{L}_3 = \mathtt{0}$.

### 5.2.3 Changes of coordinates

Let $\mathbf{u'} = \mathtt{H}\mathbf{u}$ be a change of coordinates in $\mathbb{P}^3$. If a line $\boldsymbol{\ell}$ is determined by the join of $\mathbf{u}$ and $\mathbf{v}$, its $\mathtt{L}$-matrix will change according to

$$\mathtt{M}(\mathbf{u'},\mathbf{v'}) = \mathtt{M}(\mathtt{H}\mathbf{u},\mathtt{H}\mathbf{v}) = (\mathtt{H}\mathbf{u})(\mathtt{H}\mathbf{v})^{\mathtt{T}} - (\mathtt{H}\mathbf{v})(\mathtt{H}\mathbf{u})^{\mathtt{T}} \quad (5.6)$$

$$= \mathtt{H}\mathbf{u}\mathbf{v}^{\mathtt{T}}\mathtt{H}^{\mathtt{T}} - \mathtt{H}\mathbf{v}\mathbf{u}^{\mathtt{T}}\mathtt{H}^{\mathtt{T}} = \mathtt{H}\left(\mathbf{u}\mathbf{v}^{\mathtt{T}} - \mathbf{v}\mathbf{u}^{\mathtt{T}}\right)\mathtt{H}^{\mathtt{T}} = \mathtt{H}\mathtt{M}(\mathbf{u},\mathbf{v})\mathtt{H}^{\mathtt{T}} \quad (5.7)$$

In a similar way, planes $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ when transformed by a (point) homography $\mathtt{H}$ lead to an $\mathtt{L}^*$-matrix of the form

$$\mathtt{M}(\boldsymbol{\alpha}',\boldsymbol{\beta}') = \mathtt{M}\left(\mathtt{H}^{-\mathtt{T}}\boldsymbol{\alpha}',\mathtt{H}^{-\mathtt{T}}\boldsymbol{\beta}'\right) = \mathtt{H}^{-\mathtt{T}}\mathtt{M}(\boldsymbol{\alpha},\boldsymbol{\beta})\mathtt{H}^{-1} \quad (5.8)$$

A point homography $\mathtt{H}$ acts on $\mathtt{M}^*$-matrices as

$$(\mathtt{H}\mathbf{x})^{\mathtt{T}}\mathtt{M}^*(\mathtt{H}\mathbf{u},\mathtt{H}\mathbf{v})(\mathtt{H}\mathbf{y}) = \det(\mathtt{H}\mathbf{x},\mathtt{H}\mathbf{u},\mathtt{H}\mathbf{v},\mathtt{H}\mathbf{y}) = \det(\mathtt{H})\det(\mathbf{x},\mathbf{u},\mathbf{v},\mathbf{y}) \quad (5.9)$$

where $\mathbf{x}$ and $\mathbf{y}$ are arbitrary vectors in $\mathbb{C}^4$. We have seen before that $\mathtt{M}(\mathbf{u},\mathbf{v}) \sim \mathtt{M}^*(\boldsymbol{\alpha},\boldsymbol{\beta})$ and that $\mathtt{M}(\boldsymbol{\alpha},\boldsymbol{\beta}) \sim \mathtt{M}^*(\mathbf{u},\mathbf{v})$ and thus

$$\mathtt{M}^*(\mathtt{H}\mathbf{u},\mathtt{H}\mathbf{v}) = \lambda\mathtt{H}^{-\mathtt{T}}\mathtt{M}(\boldsymbol{\alpha},\boldsymbol{\beta})\mathtt{H}^{-1} = \lambda\mu\mathtt{H}^{-\mathtt{T}}\mathtt{M}^*(\mathbf{u},\mathbf{v})\mathtt{H}^{-1} \quad (5.10)$$

for some non-zero scale factors $\lambda$ and $\mu$. Consequently, from 5.9 we arrive at

$$(\mathtt{H}\mathbf{x})^{\mathtt{T}}\mathtt{M}^*(\mathtt{H}\mathbf{u},\mathtt{H}\mathbf{v})(\mathtt{H}\mathbf{y}) = (\mathtt{H}\mathbf{x})^{\mathtt{T}}\left(\rho\mathtt{H}^{-\mathtt{T}}\mathtt{M}^*(\mathbf{u},\mathbf{v})\mathtt{H}^{-1}\right)(\mathtt{H}\mathbf{y}) \quad (5.11)$$

$$= \rho\mathbf{x}^{\mathtt{T}}\mathtt{M}^*(\mathbf{u},\mathbf{v})\mathbf{y} = \rho\det(\mathbf{x},\mathbf{u},\mathbf{v},\mathbf{y}) \quad (5.12)$$

where $\rho = \det(\mathtt{H})$. Thus,

$$\mathtt{M}^*(\mathtt{H}\mathbf{u},\mathtt{H}\mathbf{v}) = \det(\mathtt{H})\,\mathtt{H}^{-\mathtt{T}}\mathtt{M}^*(\mathbf{u},\mathbf{v})\mathtt{H}^{-1} \quad (5.13)$$

## 5.3 Plücker coordinates

For reasons that will become apparent further in the text, we may choose the following basis for the set of $4 \times 4$ antisymmetric matrices:

$$\begin{aligned}\mathtt{B} &= \{\mathtt{M}(\mathbf{e_3},\mathbf{e_4}),\mathtt{M}(\mathbf{e_1},\mathbf{e_4}),\mathtt{M}(\mathbf{e_2},\mathbf{e_4}),\mathtt{M}(\mathbf{e_3},\mathbf{e_1}),\mathtt{M}(\mathbf{e_3},\mathbf{e_2}),\mathtt{M}(\mathbf{e_1},\mathbf{e_2})\}\\ &= \{\mathtt{M}^*(\mathbf{e_1},\mathbf{e_2}),\mathtt{M}^*(\mathbf{e_2},\mathbf{e_3}),\mathtt{M}^*(\mathbf{e_3},\mathbf{e_1}),\mathtt{M}^*(\mathbf{e_2},\mathbf{e_4}),\mathtt{M}^*(\mathbf{e_1},\mathbf{e_4}),\mathtt{M}^*(\mathbf{e_3},\mathbf{e_4})\}\end{aligned} \tag{5.14}$$

where $\{\mathbf{e}_i \in \mathbb{C}^4 \,|\, 1 \le i \le 4\}$ is the canonical basis of $\mathbb{C}^4$. With this choice of basis, an antisymmetric matrix $\mathtt{A} = (a_{ij}) \in \mathbb{C}^{4\times4}$ may be written as a vector

$$\boldsymbol{\ell}_\mathtt{A} = (a_{34}, a_{14}, a_{24}, a_{31}, a_{23}, a_{12})^\mathtt{T}$$

**Definition 5.** The *Plücker coordinates* of a line are the coordinates of its $\mathtt{L}$-matrix with respect to the base $\mathtt{B}$.

Let $\mathtt{M}(\mathbf{p},\mathbf{q})$ be the $\mathtt{L}$-matrix of a line $\boldsymbol{\ell}$ given by the points $\mathbf{p},\mathbf{q}$ and let $\mathtt{M}^*(\boldsymbol{\alpha},\boldsymbol{\beta})$ be the $\mathtt{L}^*$-matrix of $\boldsymbol{\ell}$ given by the planes $\boldsymbol{\alpha},\boldsymbol{\beta}$. From $\mathtt{M}(\mathbf{p},\mathbf{q}) \sim \mathtt{M}^*(\boldsymbol{\alpha},\boldsymbol{\beta})$ we get

$$\boldsymbol{\ell} \sim \boldsymbol{\ell}_{\mathtt{M}(\mathbf{p},\mathbf{q})} \sim \boldsymbol{\ell}_{\mathtt{M}^*(\boldsymbol{\alpha},\boldsymbol{\beta})}$$

The rationale behind the particular choice of basis 5.14 comes from the fact that, given a $4 \times 4$ antisymmetric matrix $\mathtt{M}$, the Plücker coordinates of its star, $\mathtt{M}^*$, can be easily computed from the coordinates of $\boldsymbol{\ell}_\mathtt{M}$ as follows

$$\boldsymbol{\ell}_{\mathtt{M}^*} = \Omega\boldsymbol{\ell}_\mathtt{M} = \begin{pmatrix} &&&&&1\\ &&&&1&\\ &&&1&&\\ &&1&&&\\ &1&&&&\\ 1&&&&& \end{pmatrix}\boldsymbol{\ell}_\mathtt{M} \tag{5.15}$$

It is immediate to realize that

$$\Omega^{-1} = \Omega \tag{5.16}$$

Another formula that will let us translate relations from Plücker matrices to Plücker coordinates is

$$\frac{1}{2}\,\mathrm{trace}(\mathtt{A}^\mathtt{T}\mathtt{B}) = \boldsymbol{\ell}_\mathtt{A}^\mathtt{T}\boldsymbol{\ell}_\mathtt{B} \tag{5.17}$$

As a consequence of this and of 5.2.2, a point in $\mathbb{P}^5$ will represent a line in $\mathbb{P}^3$ if and only if

$$\boldsymbol{\ell}_\mathtt{L}^\mathtt{T}\Omega\boldsymbol{\ell}_\mathtt{L} = 0 \tag{5.18}$$

where $\Omega$, as defined in equation 5.15, is also known as the Klein quadric.

The condition for two lines to intersect explained in Proposition 5.2.2 can be translated in terms of Plücker coordinates as

$$\frac{1}{2}\,\mathrm{trace}(\mathtt{L}_1^{\mathtt{T}}\mathtt{L}_2^*) = \boldsymbol{\ell}_{\mathtt{L}_1}{}^{\mathtt{T}}\Omega\boldsymbol{\ell}_{\mathtt{L}_2} = 0 \tag{5.19}$$

Let us define the following products of two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{C}^4$,

$$\mathbf{u} \wedge \mathbf{v} \equiv \boldsymbol{\ell}_{\mathtt{M}(\mathbf{u},\mathbf{v})} = (m_{34}, m_{14}, m_{24}, m_{31}, m_{23}, m_{12})^{\mathtt{T}} \tag{5.20}$$

$$\mathbf{u} \underset{*}{\wedge} \mathbf{v} \equiv \boldsymbol{\ell}_{\mathtt{M}^*(\mathbf{u},\mathbf{v})} = (m_{12}, m_{23}, m_{31}, m_{24}, m_{14}, m_{34})^{\mathtt{T}} \tag{5.21}$$

where $m_{ij} = u_i v_j - u_j v_i$. It is straight-forward to verify that these products are antisymmetric and bilinear.

From $\mathtt{M}(\mathbf{p}, \mathbf{q}) \sim \mathtt{M}^*(\boldsymbol{\alpha}, \boldsymbol{\beta})$ we have that, whether we define a line by a pair of planes $\boldsymbol{\alpha}, \boldsymbol{\beta}$ or by a pair of points $\mathbf{p}, \mathbf{q}$, its Plücker coordinates will satisfy

$$\boldsymbol{\alpha} \wedge \boldsymbol{\beta} \sim \mathbf{p} \underset{*}{\wedge} \mathbf{q}$$

Also, from the definition of $\Omega$ and the products above,

$$\Omega\,(\mathbf{u} \wedge \mathbf{v}) = \mathbf{u} \underset{*}{\wedge} \mathbf{v} \tag{5.22}$$

$$\Omega\left(\mathbf{u} \underset{*}{\wedge} \mathbf{v}\right) = \mathbf{u} \wedge \mathbf{v} \tag{5.23}$$

### 5.3.1 Changes of coordinates

A change of coordinates in $\mathbb{P}^3$ given by $\mathbf{p'} = \mathtt{H}\mathbf{p}$ acts on $\mathbb{P}^5$ as

$$\boldsymbol{\ell}_{\mathtt{M}(\mathbf{p'},\mathbf{q'})} = \boldsymbol{\ell}_{\mathtt{M}(\mathtt{H}\mathbf{p},\mathtt{H}\mathbf{q})} = \boldsymbol{\ell}_{\mathtt{H}\mathtt{M}(\mathbf{p},\mathbf{q})\mathtt{H}^{\mathtt{T}}} = \widetilde{\mathtt{H}}\,\boldsymbol{\ell}_{\mathtt{M}(\mathbf{p},\mathbf{q})} \tag{5.24}$$

Note that this is indeed the definition of $\widetilde{\mathtt{H}}$ because we give the mapping between elements of the basis $\mathtt{B}$. From this definition, it is clear that $\widetilde{\mathtt{H}}^{-1} = \widetilde{\mathtt{H}^{-1}}$.

We can retrieve the columns of $\widetilde{\mathtt{H}}$ by proceeding as follows. If the matrix corresponding to the space homography is written in terms of its column vectors as $\mathtt{H} = (\mathbf{h}_1, \ldots, \mathbf{h}_4)$, then

$$\mathbf{h}_i \wedge \mathbf{h}_j = \boldsymbol{\ell}_{\mathtt{M}(\mathbf{h}_i,\mathbf{h}_j)} = \boldsymbol{\ell}_{\mathtt{M}(\mathtt{H}\mathbf{e}_i,\mathtt{M}\mathbf{e}_j)} = \widetilde{\mathtt{H}}\,\boldsymbol{\ell}_{\mathtt{M}(\mathbf{e}_i,\mathbf{e}_j)}$$

This expression together with that of the basis $\mathtt{B}$ allows us to obtain the columns of $\widetilde{\mathtt{H}}$ and write it as

$$\widetilde{\mathtt{H}} = (\mathbf{h}_3 \wedge \mathbf{h}_4,\ \mathbf{h}_1 \wedge \mathbf{h}_4,\ \mathbf{h}_2 \wedge \mathbf{h}_4,\ \mathbf{h}_3 \wedge \mathbf{h}_1,\ \mathbf{h}_2 \wedge \mathbf{h}_3,\ \mathbf{h}_1 \wedge \mathbf{h}_2) \tag{5.25}$$

Matrices of this form must leave the quadric $\Omega$ invariant since they transform Plücker coordinates into Plücker coordinates. The entries of $\widetilde{\mathtt{H}}^{\mathtt{T}}\Omega\widetilde{\mathtt{H}}$ are of the form

$$(\mathbf{h}_i \wedge \mathbf{h}_j)^{\mathtt{T}}\,\Omega\,(\mathbf{h}_k \wedge \mathbf{h}_l) = \boldsymbol{\ell}^{\mathtt{T}}_{\mathtt{M}(\mathbf{h_i},\mathbf{h_j})}\,\Omega\,\boldsymbol{\ell}_{\mathtt{M}(\mathbf{h_k},\mathbf{h_l})}$$
$$= \frac{1}{2}\,\mathrm{trace}(\mathtt{M}(\mathbf{h_i},\mathbf{h_j})^{\mathtt{T}}\,\mathtt{M}^*(\mathbf{h_k},\mathbf{h_l})) = \det(\mathbf{h_i},\mathbf{h_j},\mathbf{h_k},\mathbf{h_l})$$

where the last equality comes from the proof of Proposition 5.2.2. This means that

$$\widetilde{\mathtt{H}}^{\mathtt{T}}\Omega\widetilde{\mathtt{H}} = \det(\mathtt{H})\Omega \tag{5.26}$$

Instead of $\mathtt{M}$-matrices we can use $\mathtt{M}^*$-matrices to establish a similar formula for $\widetilde{\mathtt{H}}$. Consider the fact that planes in $\mathbb{P}^3$ are transformed as $\boldsymbol{\alpha}' = \mathtt{H}^{-\mathtt{T}}\boldsymbol{\alpha}$, thus, by 5.13

$$\mathtt{M}^*(\boldsymbol{\alpha}',\boldsymbol{\beta}') = \mathtt{M}^*(\mathtt{H}^{-\mathtt{T}}\boldsymbol{\alpha},\mathtt{H}^{-\mathtt{T}}\boldsymbol{\beta}) = \frac{1}{\det(\mathtt{H})}\mathtt{H}\mathtt{M}^*(\boldsymbol{\alpha},\boldsymbol{\beta})\mathtt{H}^{\mathtt{T}}$$

and, accordingly

$$\boldsymbol{\ell}_{\mathtt{M}^*(\boldsymbol{\alpha}',\boldsymbol{\beta}')} = \frac{1}{\det(\mathtt{H})}\boldsymbol{\ell}_{\mathtt{H}\mathtt{M}^*(\boldsymbol{\alpha},\boldsymbol{\beta})\mathtt{H}^{\mathtt{T}}} = \frac{1}{\det(\mathtt{H})}\widetilde{\mathtt{H}}\,\boldsymbol{\ell}_{\mathtt{M}^*(\boldsymbol{\alpha},\boldsymbol{\beta})}$$

If we define $\widehat{\mathtt{H}} = \frac{1}{\det(\mathtt{H})}\widetilde{\mathtt{H}}$ then we can write

$$\boldsymbol{\ell}_{\mathtt{M}^*(\boldsymbol{\alpha},\boldsymbol{\beta})} = \boldsymbol{\ell}_{\mathtt{M}^*(\mathtt{H}^{\mathtt{T}}\boldsymbol{\alpha}',\mathtt{H}^{\mathtt{T}}\boldsymbol{\beta}')} = \widehat{\mathtt{H}}^{-1}\boldsymbol{\ell}_{\mathtt{M}^*(\boldsymbol{\alpha}',\boldsymbol{\beta}')}$$

Reproducing the same steps that led us to 5.25 (but taking into account the basis $\mathtt{B}$ expressed in terms of $\mathtt{M}^*$-matrices) yields

$$\widehat{\mathtt{H}}^{-1} = \Big(\mathbf{r}_1 \underset{*}{\wedge} \mathbf{r}_2,\ \mathbf{r}_2 \underset{*}{\wedge} \mathbf{r}_3,\ \mathbf{r}_3 \underset{*}{\wedge} \mathbf{r}_1,\ \mathbf{r}_2 \underset{*}{\wedge} \mathbf{r}_4,\ \mathbf{r}_1 \underset{*}{\wedge} \mathbf{r}_4,\ \mathbf{r}_3 \underset{*}{\wedge} \mathbf{r}_4\Big)$$

where $\mathbf{r}_i$ are the rows of $\mathtt{H}$. Now, by 5.26

$$\widetilde{\mathtt{H}} = \det(\mathtt{H})\,\Omega^{-1}\,\widetilde{\mathtt{H}}^{-\mathtt{T}}\,\Omega = \Omega\,\widehat{\mathtt{H}}^{-1}\,\Omega$$

and taking into account that right multiplying a matrix by $\Omega$ amounts to inverting the order of its columns, we obtain the expression

$$\widetilde{\mathtt{H}}^{\mathtt{T}} = (\mathbf{r}_3 \wedge \mathbf{r}_4,\ \mathbf{r}_1 \wedge \mathbf{r}_4,\ \mathbf{r}_2 \wedge \mathbf{r}_4,\ \mathbf{r}_3 \wedge \mathbf{r}_1,\ \mathbf{r}_2 \wedge \mathbf{r}_3,\ \mathbf{r}_1 \wedge \mathbf{r}_2) = \widetilde{\mathtt{H}}^{\mathtt{T}} \tag{5.27}$$

## 5.4 The absolute quadratic complex

### 5.4.1 Introduction

We introduced the absolute dual quadric $\mathtt{Q}^*_\infty$ in chapter 3, a geometric entity that conveys the same information as the absolute conic but has the advantage of being easily expressed in matrix notation. In this section we are going to develop the theory around the *absolute quadratic complex*, another geometric object that is equivalent to the absolute conic. While the absolute dual quadric was defined by the tangent planes to the absolute conic, the absolute quadratic complex is defined by the secant lines to the absolute conic and shares with the absolute dual quadric its ease of use from a computational standpoint.

The absolute dual quadric $\mathtt{Q}^*_\infty$ is a correlation that assigns to a plane $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)^\mathtt{T}$ a point $\mathbf{x} = \mathtt{Q}^*_\infty\boldsymbol{\alpha}$ lying in $\boldsymbol{\pi}_\infty$ such that $\mathbf{x}$ is the orthogonal direction to $\boldsymbol{\alpha}$. In order to prove this, it suffices to establish the result in a Euclidean reference frame which immediately leaves us with

$$\mathbf{x} = (\mathtt{Q}^*_\infty)^{\mathrm{euc}}\boldsymbol{\alpha} = \begin{pmatrix} \mathtt{I}_3 & \mathbf{0} \\ \mathbf{0}^\mathtt{T} & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ 0 \end{pmatrix}$$

So it is clear that $\mathbf{x} \in \boldsymbol{\pi}_\infty$ and is an orthogonal direction to the plane $\boldsymbol{\alpha}$ whose equation is $\sum_{i=1}^4 \alpha_i x_i = 0$.

If a line $\boldsymbol{\ell}$ given by two planes $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ is not contained in the plane at infinity then the line $\boldsymbol{\ell}^\perp$ joining points $\mathtt{Q}^*_\infty\boldsymbol{\alpha}$ and $\mathtt{Q}^*_\infty\boldsymbol{\beta}$ is the range of orthogonal directions to $\boldsymbol{\ell}$. The $\mathtt{L}$-matrix associated to $\boldsymbol{\ell}^\perp$ is

$$\mathtt{L}^\perp = \mathtt{M}(\mathtt{Q}^*_\infty\boldsymbol{\alpha}, \mathtt{Q}^*_\infty\boldsymbol{\beta}), \tag{5.28}$$

and the $\mathtt{L}^*$-matrix associated to $\boldsymbol{\ell}$ is

$$\mathtt{L}^* = \mathtt{M}(\boldsymbol{\alpha}, \boldsymbol{\beta}). \tag{5.29}$$

By the properties of $\mathtt{L}$-matrices we arrive at

$$\mathtt{L}^\perp = \mathtt{Q}^*_\infty \mathtt{L}^* (\mathtt{Q}^*_\infty)^\mathtt{T} = \mathtt{Q}^*_\infty \mathtt{L}^* \mathtt{Q}^*_\infty \tag{5.30}$$

Note that the definition of $\boldsymbol{\ell}^\perp$ makes sense even in the case of $\boldsymbol{\ell}$ not being real. However, if $\boldsymbol{\ell}$ is contained in the plane at infinity then we can take it as one of its defining planes and noting that $\mathtt{Q}^*_\infty\boldsymbol{\pi}_\infty = \mathbf{0}$, we have that

$$\mathtt{L}^\perp = \mathtt{M}(\mathtt{Q}^*_\infty\boldsymbol{\pi}_\infty, \mathtt{Q}^*_\infty\boldsymbol{\beta}) = \mathtt{M}(\mathbf{0}, \mathtt{Q}^*_\infty\boldsymbol{\beta}) = \mathbf{0}$$

which agrees with $\boldsymbol{\ell}^{\perp}$ not being defined for $\boldsymbol{\ell} \subset \boldsymbol{\pi}_{\infty}$.

In light of the discussion above, two coplanar lines $\boldsymbol{\ell}$ and $\boldsymbol{\ell}'$ are orthogonal if and only if one of the orthogonal directions of $\boldsymbol{\ell}$ coincides with the point at infinity of $\boldsymbol{\ell}'$. This notion can be succinctly expressed using Plücker matrices as

$$0 = \mathrm{trace}(\mathtt{L}^{*\prime}\mathtt{L}^{\perp}) = \mathrm{trace}(\mathtt{L}^{*\prime}\mathtt{Q}^{*}_{\infty}\mathtt{L}^{*}\mathtt{Q}^{*}_{\infty}) \tag{5.31}$$

If we denote by $\mathbf{p}_{\infty}$ the point where $\boldsymbol{\ell}$ meets the plane at infinity, it turns out that $\boldsymbol{\ell}^{\perp}$ is the polar line of $\mathbf{p}_{\infty}$ with respect to the absolute conic. Thus, a line $\boldsymbol{\ell}$ not contained in $\boldsymbol{\pi}_{\infty}$ intersects the absolute conic if and only if it intersects $\boldsymbol{\ell}^{\perp}$. Those lines are the *isotropic lines* and they are characterized by the relation

$$0 = \mathrm{trace}(\mathtt{L}^{*}\mathtt{Q}^{*}_{\infty}\mathtt{L}^{*}\mathtt{Q}^{*}_{\infty}) = \mathrm{trace}\left[(\mathtt{L}^{*}\mathtt{Q}^{*}_{\infty})^{2}\right] \tag{5.32}$$

This quadratic expression in the entries of $\mathtt{L}^{*}$ is called the *absolute quadratic complex* (abbreviated AQC) and it allows us to endow $\mathbb{P}^{3}$ with Euclidean geometry as will be demonstrated further in the text.

### 5.4.2 The AQC in Plücker coordinates.

Due to being defined in terms of the trace, expression 5.31 is a symmetric bilinear map in the entries of $\mathtt{L}^{*}$ and $\mathtt{L}^{*\prime}$ and so it can be represented by a $6 \times 6$ symmetric matrix $\Sigma$ such that

$$\frac{1}{2}\,\mathrm{trace}\left[\left(\mathtt{L}^{*\prime}\right)^{\mathtt{T}}\mathtt{Q}^{*}_{\infty}\mathtt{L}^{*}\mathtt{Q}^{*}_{\infty}\right] = \boldsymbol{\ell}^{\mathtt{T}}_{\mathtt{L}'}\,\Sigma\,\boldsymbol{\ell}_{\mathtt{L}} \tag{5.33}$$

This means that we can restate the condition for two coplanar lines (not lying in the plane at infinity) to be orthogonal as

$$\boldsymbol{\ell}'^{\mathtt{T}}\,\Sigma\,\boldsymbol{\ell} = 0 \tag{5.34}$$

Thus, a line $\boldsymbol{\ell}$ is isotropic if and only if $\boldsymbol{\ell}^{\mathtt{T}}\,\Sigma\,\boldsymbol{\ell} = 0$.

Let us take a closer look at relation 5.33. For every $\boldsymbol{\ell}_{\mathtt{L}}, \boldsymbol{\ell}_{\mathtt{L}'} \in \mathbb{P}^{5}$ we have, on one hand

$$\frac{1}{2}\,\mathrm{trace}\left[\left(\mathtt{L}^{*\prime}\right)^{\mathtt{T}}\mathtt{Q}^{*}_{\infty}\mathtt{L}^{*}\mathtt{Q}^{*}_{\infty}\right] = \boldsymbol{\ell}^{\mathtt{T}}_{\mathtt{L}'}\,\Sigma\,\boldsymbol{\ell}_{\mathtt{L}} = \boldsymbol{\ell}^{\mathtt{T}}_{\mathtt{L}'}\,\Omega\,(\Omega\,\Sigma\,\boldsymbol{\ell}_{\mathtt{L}})$$

and on the other hand (note that $\boldsymbol{\ell}^{\mathtt{T}}_{\mathtt{L}^{*\prime}} = \boldsymbol{\ell}^{\mathtt{T}}_{\mathtt{L}^{*\prime}}\,\Omega^{2} = \boldsymbol{\ell}^{\mathtt{T}}_{\mathtt{L}'}\,\Omega$)

$$\begin{aligned}\frac{1}{2}\,\mathrm{trace}\left[\left(\mathtt{L}^{*\prime}\right)^{\mathtt{T}}\mathtt{Q}^{*}_{\infty}\mathtt{L}^{*}\mathtt{Q}^{*}_{\infty}\right] &= \boldsymbol{\ell}^{\mathtt{T}}_{\mathtt{L}^{*\prime}}\,\boldsymbol{\ell}_{\mathtt{Q}^{*}_{\infty}\mathtt{L}^{*}\mathtt{Q}^{*}_{\infty}} = \boldsymbol{\ell}^{\mathtt{T}}_{\mathtt{L}'}\,\Omega\,\boldsymbol{\ell}_{\mathtt{Q}^{*}_{\infty}\mathtt{L}^{*}\mathtt{Q}^{*}_{\infty}} \\ &= \boldsymbol{\ell}^{\mathtt{T}}_{\mathtt{L}'}\,\Omega\,\boldsymbol{\ell}^{\perp}_{\mathtt{L}}.\end{aligned}$$

Thus, we arrive at the identity

$$\boldsymbol{\ell}^{\perp} = \Omega\,\Sigma\,\boldsymbol{\ell} \tag{5.35}$$

from which we can immediately deduce that $\mathrm{im}(\Omega\,\Sigma) = \boldsymbol{\pi}_\infty$.

Consider the canonical basis $\{\mathbf{e_i} \in \mathbb{C}^6 \,|\, 1 \leq i \leq 6\}$. Each element of the basis represents a line because $\mathbf{e_i}^{\mathtt{T}}\,\Omega\,\mathbf{e_i} = \mathbf{e_i}^{\mathtt{T}}\,\mathbf{e_{6-i+1}} = 0$ for $1 \leq i \leq 6$ and, from 5.35, the columns $\boldsymbol{\ell}_i = \Omega\,\Sigma\,\mathbf{e_i}$ of $\Omega\,\Sigma$ are lines in $\boldsymbol{\pi}_\infty$. This fact has some interesting consequences:

1. Each of the $\boldsymbol{\ell}_i$ verify on one hand that $\boldsymbol{\ell}_i^{\mathtt{T}}\,\Omega\,\boldsymbol{\ell}_i = 0$ and on the other hand, since they all lie in the plane at infinity, $\boldsymbol{\ell}_i^{\mathtt{T}}\,\Omega\,\boldsymbol{\ell}_j = 0$ for every $i, j \in \{1, \ldots, 6\}$. This is equivalent to

$$\mathtt{0} = \begin{pmatrix} \boldsymbol{\ell}_1^{\mathtt{T}}\,\Omega \\ \vdots \\ \boldsymbol{\ell}_6^{\mathtt{T}}\,\Omega \end{pmatrix} (\boldsymbol{\ell}_1, \ldots, \boldsymbol{\ell}_6) = \Sigma^{\mathtt{T}}\,\Omega\,\Sigma = \Sigma\,\Omega\,\Sigma \tag{5.36}$$

2. Since the columns of $\Omega\,\Sigma$ span the lines contained in a plane ($\boldsymbol{\pi}_\infty$) and $\Omega$ is of full rank, then $\Sigma$ is of rank three.

3. The null space of $\Sigma$ is the set of all lines lying in $\boldsymbol{\pi}_\infty$ because the rank of $\Sigma$ is three, $\Sigma\,(\Omega\,\Sigma\,\boldsymbol{\ell}) = 0$ for each $\boldsymbol{\ell} \in \mathbb{P}^5$ (from 5.36), and $\mathrm{im}(\Omega\,\Sigma) = \boldsymbol{\pi}_\infty$.

### 5.4.3 Obtainment of the DAQ from the AQC.

As a consequence of the discussion in the previous section, for every pair of lines given by $\boldsymbol{\ell}_{\mathrm{L}}$ and $\boldsymbol{\ell}_{\mathrm{L}'}$ we have

$$\boldsymbol{\ell}_{\mathrm{L}'}^{\mathtt{T}}\,\Sigma\,\boldsymbol{\ell}_{\mathrm{L}} = \frac{1}{2}\,\mathrm{trace}\left[\left(\mathtt{L}^{*\prime}\right)^{\mathtt{T}}\mathtt{Q}_\infty^*\mathtt{L}^*\mathtt{Q}_\infty^*\right] = \boldsymbol{\ell}_{\mathrm{L}'}^{\mathtt{T}}\,\Omega\,\boldsymbol{\ell}_{\mathtt{Q}_\infty^*\mathtt{L}^*\mathtt{Q}_\infty^*}$$

Thus, the following relation

$$\Sigma\,\boldsymbol{\ell}_{\mathrm{L}} = \Omega\,\boldsymbol{\ell}_{\mathtt{Q}_\infty^*\mathtt{L}^*\mathtt{Q}_\infty^*} = \Omega\,\widetilde{\mathtt{Q}_\infty^*}\,\boldsymbol{\ell}_{\mathrm{L}^*} = \Omega\,\widetilde{\mathtt{Q}_\infty^*}\,\Omega\,\boldsymbol{\ell}_{\mathrm{L}}$$

holds for every line $\boldsymbol{\ell}_{\mathrm{L}}$. That is, $\Sigma = \Omega\,\widetilde{\mathtt{Q}_\infty^*}\,\Omega$.

If the DAQ is represented by the matrix given as column vectors $\mathtt{Q}_\infty^* = (\mathbf{q_1}, \mathbf{q_2}, \mathbf{q_3}, \mathbf{q_4})$ then, by 5.25,

$$\widetilde{\mathtt{Q}_\infty^*} = (\mathbf{q}_3 \wedge \mathbf{q}_4,\, \mathbf{q}_1 \wedge \mathbf{q}_4,\, \mathbf{q}_2 \wedge \mathbf{q}_4,\, \mathbf{q}_3 \wedge \mathbf{q}_1,\, \mathbf{q}_2 \wedge \mathbf{q}_3,\, \mathbf{q}_1 \wedge \mathbf{q}_2)$$

Computing the product

$$\Omega\,\widetilde{\mathtt{Q}^*_\infty} = \Big(\mathbf{q}_3 \underset{*}{\wedge} \mathbf{q}_4,\ \mathbf{q}_1 \underset{*}{\wedge} \mathbf{q}_4,\ \mathbf{q}_2 \underset{*}{\wedge} \mathbf{q}_4,\ \mathbf{q}_3 \underset{*}{\wedge} \mathbf{q}_1,\ \mathbf{q}_2 \underset{*}{\wedge} \mathbf{q}_3,\ \mathbf{q}_1 \underset{*}{\wedge} \mathbf{q}_2\Big)$$

yields an explicit expression for the AQC in terms of the DAQ:

$$\Sigma = \Big(\mathbf{q}_1 \underset{*}{\wedge} \mathbf{q}_2, \mathbf{q}_2 \underset{*}{\wedge} \mathbf{q}_3,\ \mathbf{q}_3 \underset{*}{\wedge} \mathbf{q}_1,\ \mathbf{q}_2 \underset{*}{\wedge} \mathbf{q}_4,\ \mathbf{q}_1 \underset{*}{\wedge} \mathbf{q}_4,\ \mathbf{q}_3 \underset{*}{\wedge} \mathbf{q}_4\Big) \tag{5.37}$$

### 5.4.4 The AQC in a Euclidean reference frame.

The matrix representing the absolute dual quadric in a Euclidean reference frame is $(\mathtt{Q}^*_\infty)^{\text{euc}} = (\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}, \mathbf{0})$. Substituting this expression in equation 5.37 gives us the matrix of the absolute quadratic complex in Euclidean coordinates

$$\Sigma^{\text{euc}} = \begin{pmatrix} \mathtt{I}_{3\times 3} & \mathtt{0}_{3\times 3} \\ \mathtt{0}_{3\times 3} & \mathtt{0}_{3\times 3} \end{pmatrix} \tag{5.38}$$

Conversely, if the matrix of the AQC is such that $\Sigma \sim \Sigma^{\text{euc}}$ then the reference frame must be Euclidean. This follows from realizing that:

1. The three lines given by the non-zero columns of $\Omega\,\Sigma$ span the lines in $\boldsymbol{\pi}_\infty$ and they all lie in the plane with coordinates $(0,0,0,1)^{\mathtt{T}}$ as can be readily checked by verifying that this vector is in the kernel of each $\mathtt{L}$-matrix corresponding to the lines. In other words, the plane at infinity is given by the equation $x_3 = 0$.

2. A line $\boldsymbol{\ell}$ joining the points $\mathbf{p} = (0,0,0,1)^{\mathtt{T}}$ with $\mathbf{q} = (x_0, x_1, x_2, 0)^{\mathtt{T}}$ belongs to the absolute quadratic complex, i.e.: is an isotropic line, if and only if

$$\boldsymbol{\ell}^{\mathtt{T}}\,\Sigma^{\text{euc}}\,\boldsymbol{\ell} = (\mathbf{p}\wedge\mathbf{q})^{\mathtt{T}}\,\Sigma^{\text{euc}}\,(\mathbf{p}\wedge\mathbf{q}) = x_0^2 + x_1^2 + x_2^2 = 0$$

which is the equation of the absolute conic in a Euclidean reference frame.

### 5.4.5 Changes of coordinates and the AQC.

We have already studied in section 5.3.1 how a change of coordinates $\mathbf{p} = \mathtt{H}\mathbf{p}'$ in $\mathbb{P}^3$ induces a change of coordinates $\boldsymbol{\ell} = \widetilde{\mathtt{H}}\,\boldsymbol{\ell}'$ in $\mathbb{P}^5$. The homography $\widetilde{\mathtt{H}}$ acts on the absolute quadratic complex as on any other quadric

$$\Sigma' = \widetilde{\mathtt{H}}^{\mathtt{T}}\,\Sigma\,\widetilde{\mathtt{H}}$$

Let $\mathbf{r_i}$ denote the row vectors of the matrix corresponding to the homography $\mathtt{H}$ (with $1 \le i \le 4$) that transforms points from an arbitrary reference frame to a Euclidean one. Using that $\widetilde{\mathtt{H}^\mathtt{T}} = \widetilde{\mathtt{H}}^\mathtt{T}$ and the characterization of the AQC matrix in a Euclidean reference frame, we have

$$\Sigma' = \widetilde{\mathtt{H}}^\mathtt{T}\,\Sigma^{\mathrm{euc}}\,\widetilde{\mathtt{H}} = \begin{pmatrix}\mathbf{r}_3\wedge\mathbf{r}_4, & \mathbf{r}_1\wedge\mathbf{r}_4, & \mathbf{r}_2\wedge\mathbf{r}_4\end{pmatrix}\begin{pmatrix}\mathbf{r}_3\wedge\mathbf{r}_4\\ \mathbf{r}_1\wedge\mathbf{r}_4\\ \mathbf{r}_2\wedge\mathbf{r}_4\end{pmatrix}$$

If we instead use $\widetilde{\mathtt{H}} = \det(\mathtt{H})\,\Omega^{-1}\,\widetilde{\mathtt{H}}^{-\mathtt{T}}\,\Omega$ and denote by $\mathbf{c_i}$ the column vectors of $\mathtt{H}^{-1}$ (with $1 \le i \le 4$) then we arrive at a different expression for $\Sigma'$, namely

$$\Sigma' = \Omega\,\widetilde{\mathtt{H}^{-\mathtt{T}}}\,\Omega = \begin{pmatrix}\mathbf{c}_1 \underset{*}{\wedge} \mathbf{c}_2, & \mathbf{c}_2 \underset{*}{\wedge} \mathbf{c}_3, & \mathbf{c}_3 \underset{*}{\wedge} \mathbf{c}_1\end{pmatrix}\begin{pmatrix}\mathbf{c}_1 \underset{*}{\wedge} \mathbf{c}_2\\ \mathbf{c}_2 \underset{*}{\wedge} \mathbf{c}_3\\ \mathbf{c}_3 \underset{*}{\wedge} \mathbf{c}_1\end{pmatrix}$$

### 5.4.6 A linear constraint on the AQC.

The matrices $\Sigma$ representing the absolute quadratic complex in arbitrary coordinates are contained in an hyperplane of Sym(6) (the vector space of $6\times 6$ symmetric matrices) since they always satisfy the following linear constraint

$$\begin{aligned}\mathrm{trace}(\Omega\;\Sigma) &\overset{5.26}{=} \mathrm{trace}(\widetilde{\mathtt{H}}^{-1}\,\Omega\,\widetilde{\mathtt{H}}^{-\mathtt{T}}\,\widetilde{\mathtt{H}}^{\mathtt{T}}\,\Sigma^{\mathrm{euc}}\,\widetilde{\mathtt{H}}) = \mathrm{trace}(\widetilde{\mathtt{H}}^{-1}\,\Omega\,\Sigma^{\mathrm{euc}}\,\widetilde{\mathtt{H}})\\ &= \mathrm{trace}(\widetilde{\mathtt{H}}\,\widetilde{\mathtt{H}}^{-1}\,\Omega\,\Sigma^{\mathrm{euc}}) = \mathrm{trace}(\Omega\,\Sigma^{\mathrm{euc}}) = 0\end{aligned}$$

### 5.4.7 Angle between two lines in terms of the AQC.

**Definition 6.** Let $\boldsymbol{\ell}$ and $\boldsymbol{\ell'}$ be two real lines. Then $\theta = \min(\phi, \pi - \phi)$ is the *angle between* $\boldsymbol{\ell}$ *and* $\boldsymbol{\ell'}$ where $\phi$ is the angle between any of the direction vectors of the lines.

**Proposition 5.4.1.** Let $\boldsymbol{\ell}$ and $\boldsymbol{\ell'}$ be the Plücker coordinates of two real lines. Then

$$\cos\theta = \frac{|\boldsymbol{\ell}^\mathtt{T}\,\Sigma^{\mathrm{euc}}\,\boldsymbol{\ell}|}{\sqrt{\left(\boldsymbol{\ell}^\mathtt{T}\,\Sigma^{\mathrm{euc}}\,\boldsymbol{\ell}\right)\left(\boldsymbol{\ell'}^\mathtt{T}\,\Sigma^{\mathrm{euc}}\,\boldsymbol{\ell'}\right)}} \tag{5.39}$$

*Proof.* Since the angle between two lines is a Euclidean invariant, it suffices to prove this in a Euclidean reference frame. If we choose the plane at infinity as

$\boldsymbol{\pi}_\infty = (0,0,0,1)^{\mathtt{T}}$ and note that we are working in terms of the basis specified in 5.14 then a line $\boldsymbol{\ell} = (l_1, \dots, l_6)^{\mathtt{T}} \in \mathbb{P}^5$ intersects $\boldsymbol{\pi}_\infty$ at the point given by

$$\begin{pmatrix} l_2 \\ l_3 \\ l_1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & l_6 & -l_4 & l_2 \\ -l_6 & 0 & l_5 & l_3 \\ l_4 & -l_5 & 0 & l_1 \\ -l_2 & -l_3 & -l_1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

and that means that the direction of an arbitrary line $\boldsymbol{\ell}$ is the vector $\mathbf{d} = (l_2, l_3, l_1)^{\mathtt{T}}$. As a consequence of this,

$$\cos\theta = \frac{\mathbf{d}^{\mathtt{T}}\mathbf{d}'}{\sqrt{\left(\mathbf{d}^{\mathtt{T}}\mathbf{d}\right)\left(\mathbf{d}'^{\mathtt{T}}\mathbf{d}'\right)}} = \frac{|\boldsymbol{\ell}^{\mathtt{T}}\,\Sigma^{\mathrm{euc}}\,\boldsymbol{\ell}|}{\sqrt{\left(\boldsymbol{\ell}^{\mathtt{T}}\,\Sigma^{\mathrm{euc}}\,\boldsymbol{\ell}\right)\left(\boldsymbol{\ell}'^{\mathtt{T}}\,\Sigma^{\mathrm{euc}}\,\boldsymbol{\ell}'\right)}}$$

□

### 5.4.8 Computing the DAQ from the AQC.

The fact that $\det(\mathbf{x}, \mathbf{u}, \mathbf{v}, \mathbf{y}) = \mathbf{x}^{\mathtt{T}}\mathtt{M}^*(\mathbf{u}, \mathbf{v})\mathbf{y}$ leads us to the following linear system of equations in the entries $\mathbf{q_i}$ of $\mathtt{Q}^*_\infty$

$$\begin{cases} \mathtt{M}^*(\mathbf{q_i}, \mathbf{q_j})\mathbf{q_i} = \mathbf{0} \\ \mathtt{M}^*(\mathbf{q_i}, \mathbf{q_j})\mathbf{q_k} - \mathtt{M}^*(\mathbf{q_k}, \mathbf{q_i})\mathbf{q_j} = \mathbf{0} \end{cases}$$

Relations 5.37 and 5.21 allow us to obtain the matrices involved in the left hand sides of this system. A solution for this system is a matrix in Sym(4) (which has 10 degrees of freedom). The DAQ matrix is rank deficient so we have to remove the extra degree of freedom by approximating the solution by the closest rank 3 matrix. This task can be accomplished by applying the *singular value decomposition* to the solution and truncating the smallest singular value to zero (see [HZ04]).

### 5.4.9 Euclidean reconstruction from the AQC.

The following result leads us to a practical way to obtain a rectifying homography from the absolute quadratic complex.

**Theorem 5.4.1.** Let $\Sigma = \mathtt{G}^{\mathtt{T}}\,\Sigma^{\mathrm{euc}}\,\mathtt{G}$ be the matrix corresponding to the AQC under a certain change of coordinates in $\mathbb{P}^3$ and let $\mathtt{G} = (\mathbf{r_1}, \dots, \mathbf{r_6})^{\mathtt{T}}$. Then there exist vectors $\mathbf{v_1}, \dots, \mathbf{v_4} \in \mathbb{C}^4$ such that $\mathtt{H}^{\mathtt{T}} = (\mathbf{v_1}, \dots, \mathbf{v_4})$ is a change of coordinates from the current reference frame to a Euclidean one.

*Proof.* Let $\mathcal{R} = (\mathbf{r_1}, \mathbf{r_2}, \mathbf{r_3}) \in \mathbb{C}^{3\times 6}$, then

$$\Sigma = \mathtt{G}^{\mathtt{T}}\, \Sigma^{\text{euc}}\, \mathtt{G} = \mathcal{R}\mathcal{R}^{\mathtt{T}}$$

and $\operatorname{rank}(\mathcal{R}) = \operatorname{rank}(\Sigma) = 3$. Noting that $\mathtt{G}^{\mathtt{T}}\, \Sigma^{\text{euc}} = (\mathcal{R} \,|\, \mathtt{0}_{3\times 6})$ we have

$$\mathtt{0} = \Sigma\,\Omega\,\Sigma = \mathtt{G}^{\mathtt{T}}\, \Sigma^{\text{euc}}\, \mathtt{G}\,\Omega\,\mathtt{G}^{\mathtt{T}}\, \Sigma^{\text{euc}}\, \mathtt{G} = \mathcal{R}\mathcal{R}^{\mathtt{T}}$$

and taking into account that $\mathtt{G}$ is regular, we see that

$$\mathcal{R}\,\Omega\,\mathcal{R}^{\mathtt{T}} = \mathtt{0}.$$

This implies that $\mathbf{r_i}^{\mathtt{T}}\,\Omega\mathbf{r_j}$ for $i, j \in \{1, 2, 3\}$ and, in particular, that the $\mathbf{r_i}$ are Plücker coordinates of lines in $\mathbb{P}^3$ .

Let $\mathbf{s} \in \mathbb{P}^5$ be any line intersecting the $\mathbf{r_1}, \mathbf{r_2}, \mathbf{r_3}$. Then $\mathbf{r_i}^{\mathtt{T}}\,\Omega\,\mathbf{s} = 0$ for each $i \in \{1, 2, 3\}$. So $\mathcal{R}^{\mathtt{T}}\,\Omega\,\mathbf{s} = 0$ and, consequently,

$$(\Sigma\,\Omega)\,\mathbf{s} = \mathcal{R}\,\mathcal{R}^{\mathtt{T}}\,\Omega\,\mathbf{s} = 0$$

From this, it is clear that the set of lines intersecting the $\mathbf{r_i}$ is contained in the null space of $\Sigma\,\Omega$. Since the dimension of this set is the same as the dimension of the null space of $\Sigma\,\Omega$ we conclude that both spaces are the same.

Observe that $\ker(\Sigma\,\Omega) = \Omega\,\ker(\Sigma)$, for $\boldsymbol{\ell} \in \ker(\Sigma)$ if and only if $\Sigma\,\boldsymbol{\ell} = (\Sigma\,\Omega)\,(\Omega\,\boldsymbol{\ell}) = 0$. Since $\ker(\Sigma)$ is the set of lines contained in a plane, $\ker(\Sigma\,\Omega) = \Omega\,\ker(\Sigma)$ is a star of lines meeting in a point. Thus, the lines intersecting $\mathbf{r_1}, \mathbf{r_2}, \mathbf{r_3}$ meet in a point which implies that there is a common point given by $\mathbf{v_4}$ in which all the $\mathbf{r_i}$ meet. If we take $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3} \in \mathbb{C}^4$ such that

$$\begin{aligned} \mathbf{r_1} &= \mathbf{v}_3 \wedge \mathbf{v}_4 \\ \mathbf{r_2} &= \mathbf{v}_1 \wedge \mathbf{v}_4 \\ \mathbf{r_3} &= \mathbf{v}_2 \wedge \mathbf{v}_4 \end{aligned}$$

then

$$\Sigma = \mathcal{R}\mathcal{R}^{\mathtt{T}} = \big(\mathbf{v}_3 \wedge \mathbf{v}_4,\ \mathbf{v}_1 \wedge \mathbf{v}_4,\ \mathbf{v}_2 \wedge \mathbf{v}_4\big) \begin{pmatrix} \mathbf{v}_3 \wedge \mathbf{v}_4 \\ \mathbf{v}_1 \wedge \mathbf{v}_4 \\ \mathbf{v}_2 \wedge \mathbf{v}_4 \end{pmatrix} = \widetilde{\mathtt{H}}^{\mathtt{T}}\, \Sigma^{\text{euc}}\, \widetilde{\mathtt{H}}$$

and so $\mathtt{H}^{\mathtt{T}} = (\mathbf{v_1}, \ldots, \mathbf{v_4})$ is the rectifying homography. □

# Chapter 6

# The quadric of segments.

We will now introduce a geometric entity known as the *quadric of segments* which provides a means of performing a Euclidean upgrading of a projective reconstruction using the knowledge of lengths of several segments in a scene.

## 6.1 Introduction.

Let $d \in \mathbb{R}_{\geq 0}$ and $\mathbf{X} = (x_1, \ldots, x_n)^{\mathsf{T}}$, $\mathbf{Y} = (y_1, \ldots, y_n)^{\mathsf{T}} \in \mathbb{C}^n$. Consider the equation of a sphere of radius $d$ centered at a point $\mathbf{Y}$.

$$(\mathbf{X} - \mathbf{Y})^{\mathsf{T}}(\mathbf{X} - \mathbf{Y}) = \sum_{i=1}^{n} (X_i - Y_i)^2 = d^2 \tag{6.1}$$

Note that the segments of length $d$ having coordinates in $\mathbb{R}$ are among the roots of this equation.

Choosing $x_0 = 0$ as the equation for the hyperplane at infinity in $\mathbb{P}^n$ and homogenizing equation 6.1 ($X_i = \frac{x_i}{x_0}, Y_i = \frac{y_i}{y_0}$), we get

$$\sum_{i=1}^{n} \left( \frac{x_i}{x_0} - \frac{y_i}{y_0} \right)^2 = \sum_{i=1}^{n} \frac{(x_i y_0 - y_i x_0)^2}{{x_0}^2 {y_0}^2} = d^2$$

and finally arrive at

$$\sum_{i=1}^{n} (x_i y_0 - y_i x_0)^2 - d^2 {x_0}^2 {y_0}^2 = 0 \tag{6.2}$$

We will denote by $\mathcal{V}_d$ the algebraic variety defined in $\mathbb{P}^n \times \mathbb{P}^n$ by equation 6.2.

*Remark* 6.1.1. Setting $d = 0$ and fixing $\mathbf{Y}$ in equation 6.1 leaves us with a quadric whose projective completion can be expressed (after a translation) by the equation $\sum_{i=1}^{n} x_i^2 = 0$. This quadric intersects the hyperplane at infinity, given by $x_0 = 0$, in the absolute conic.

Consider two points $\mathbf{x}, \mathbf{y} \in \mathcal{V}_d$ such that $\mathbf{x} \in \boldsymbol{\pi}_\infty$ and $\mathbf{y} \notin \boldsymbol{\pi}_\infty$. For these points equation 6.2 is left as

$$\begin{cases} \sum_{i=1}^{n} x_i^2 = 0 \\ x_0 = 0 \end{cases}$$

*Remark* 6.1.2. Radius zero spheres given by the equation

$$\sum_{i=1}^{n} (x_i y_0 - y_i x_0)^2 = 0$$

are quadric cones (remember that $\mathbb{C}$ is our base field) with vertex $\mathbf{y}$ and having the absolute quadric as its base in the hyperplane at infinity. If $\mathbf{x}$ and $\mathbf{y}$ are points in the radius zero sphere then any point $\mathbf{x} + \lambda \mathbf{y}$ (with $\lambda \in \mathbb{C}$) along the line joining them lies also on the sphere because

$$\begin{aligned} \sum_{i=1}^{n} \left[(x_i + \lambda y_i)\, y_0 - y_i\,(x_0 + \lambda y_0)\right]^2 &= \sum_{i=1}^{n} (x_i y_0 + \lambda y_i y_0 - y_i x_0 + \lambda y_i y_0)^2 \\ &= \sum_{i=1}^{n} (x_i y_0 - y_i x_0)^2 = 0 \end{aligned}$$

## 6.2 Geometric representation of segments.

We will now study how a certain way of representing non-oriented segments will allow us to see $\mathcal{V}_d$ as a quadric in some projective space.

Let's recall that an (unordered) pair of points $\{\mathbf{u}, \mathbf{v}\} \subset \mathbb{P}^n$ is a degenerate quadric and that its dual quadric has a matrix representation of the form

$$\mathtt{S} = \mathtt{S}(\mathbf{u}, \mathbf{v}) = \mathbf{u}\mathbf{v}^{\mathtt{T}} + \mathbf{v}\mathbf{u}^{\mathtt{T}} \tag{6.3}$$

with the entries of $\mathtt{S}$ being $\mathtt{S}_{ij} = x_i y_j + y_i x_j$ for each $i, j \in \{0, \ldots, n\}$.

The rank of the quadric given by 6.3 is at most two. This is easily deduced by inspecting the matrix resulting from the expansion of the products

$$\mathbf{u}\mathbf{v}^{\mathtt{T}} = \mathbf{u}(v_0, \ldots, v_n) = (v_0\mathbf{u}, \ldots, v_n\mathbf{u})$$

where we see that each term of the sum in 6.3 is a linear combination of $\mathbf{u}$ and $\mathbf{v}$ respectively. Consequently, $\mathtt{S}$ will be of rank two when both points are different and of rank one when they are the same.

**Theorem 6.2.1.** Let $\mathbf{x}$ and $\mathbf{y}$ be points in $\mathbb{P}^n$. There's a bijection between the set of unordered point pairs $\{\mathbf{x}, \mathbf{y}\}$ and the set of symmetric matrices $\mathtt{S}$ of ranks $\leq 2$ representing linear maps in $\mathbb{P}^n$.

*Proof.* We have already seen how we can regard $\mathtt{S}(\mathbf{u}, \mathbf{v})$ as a mapping that assigns a $(n+1) \times (n+1)$ projective symmetric matrix to each pair of points $\mathbf{u}, \mathbf{v}$.

Let's see how to obtain the inverse mapping for the case of $\mathtt{S}$ having rank two (the case with rank one is analogous and simpler). Computing the eigenvalue decomposition of $\mathtt{S}$ and taking into account that we are working with matrices defined over the complex field, then

$$\mathtt{S} = \mathtt{H}^{\mathtt{T}} \operatorname{diag}(1, 1, 0, \ldots, 0)\mathtt{H}$$

for some homography $\mathtt{H}$. We will now prove that

$$\mathbf{x} = \mathtt{H}\mathbf{v} = \mathtt{H}\begin{pmatrix} 1 \\ i \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \mathbf{y} = \mathtt{H}\mathbf{w} = \mathtt{H}\begin{pmatrix} 1 \\ -i \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

It is a straight-forward computation to see that

$$\mathbf{v}\mathbf{w}^{\mathtt{T}} + \mathbf{w}\mathbf{v}^{\mathtt{T}} = \begin{pmatrix} 1 & -i & 0 & \cdots & 0 \\ i & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} + \begin{pmatrix} 1 & i & 0 & \cdots & 0 \\ -i & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

results in $\mathbf{v}\mathbf{w}^{\mathtt{T}} + \mathbf{w}\mathbf{v}^{\mathtt{T}} \sim \operatorname{diag}(1, 1, 0, \ldots, 0)$ and, from this,

$$\mathtt{H}^{\mathtt{T}} \left(\mathbf{v}\mathbf{w}^{\mathtt{T}} + \mathbf{w}\mathbf{v}^{\mathtt{T}}\right) \mathtt{H} = (\mathtt{H}\mathbf{v})(\mathtt{H}\mathbf{w})^{\mathtt{T}} + (\mathtt{H}\mathbf{w})(\mathtt{H}\mathbf{v})^{\mathtt{T}} = \mathbf{x}\mathbf{y}^{\mathtt{T}} + \mathbf{y}\mathbf{x}^{\mathtt{T}} = \mathtt{S}$$

Let $\mathbf{v}' = (v'_0, \ldots, v'_n)^{\mathtt{T}}$ and $\mathbf{w}' = (w'_0, \ldots, w'_n)^{\mathtt{T}}$ be two vectors such that $\mathbf{v}'\mathbf{w}'^{\mathtt{T}} + \mathbf{w}'\mathbf{v}'^{\mathtt{T}} = \operatorname{diag}(1, 1, 0, \ldots, 0)$. Solving the polynomial system of equations

$$\begin{cases} 2v'_0 w'_0 = 1 \\ v'_0 w'_1 + v'_1 w'_0 = 0 \\ 2v'_1 w'_1 = 1 \end{cases}$$

leads us to the fact that $\mathbf{v}' \sim \mathbf{v}$ and $\mathbf{w}' \sim \mathbf{w}$. □

### 6.2.1 Segments as points in a projective space.

Let $\mathtt{A} = (z_{ij}) \in \mathbb{C}^{(n+1)\times(n+1)}$ be a symmetric matrix. Consider the map

$$\begin{array}{rcl} \nu : \mathrm{Sym}(n+1) & \longrightarrow & \mathbb{C}^{N+1} \\ \mathtt{A} & \mapsto & \nu(\mathtt{A}) \end{array}$$

where $N = \frac{(n+1)(n+2)}{2} - 1$ and

$$\nu(\mathtt{A}) = \left( \frac{z_{00}}{\sqrt{2}}, \frac{z_{11}}{\sqrt{2}}, \dots, \frac{z_{nn}}{\sqrt{2}}, z_{01}, z_{02}, \dots, z_{0n}, z_{12}, \dots, z_{1n}, z_{23}, \dots, z_{2n}, \dots, z_{n-1,n} \right) \tag{6.4}$$

This map $\nu$ is a coordinate system in the space of symmetric matrices of order $n+1$ and it is also a linear isometry with respect to the scalar product given by $\langle \mathtt{A}, \mathtt{B} \rangle = \frac{1}{2}\,\mathrm{trace}(\mathtt{AB})$, that is,

$$\langle \nu(\mathtt{A}), \nu(\mathtt{B}) \rangle = \nu(\mathtt{A})^{\mathtt{T}} \nu(\mathtt{B}) = \frac{1}{2}\,\mathrm{trace}(\mathtt{AB}) = \langle \mathtt{A}, \mathtt{B} \rangle$$

We will associate an unordered segment in $\mathbb{P}^n$ to a point in $\mathbb{P}^N$ via the map

$$\begin{array}{rcl} \sigma : \mathbb{P}^n \times \mathbb{P}^n & \longrightarrow & \mathbb{P}^N \\ (\mathbf{x}, \mathbf{y}) & \mapsto & \sigma(\mathbf{x}, \mathbf{y}) = \nu\left(\mathtt{S}(\mathbf{x}, \mathbf{y})\right) \end{array}$$

## 6.3 The quadric of segments.

We can write the equation 6.2 of the variety $\mathcal{V}_d$ in terms of the entries of $\mathtt{S}(\mathbf{x}, \mathbf{y}) = (x_i y_j + y_i x_j)_{i,j=1}^n = (z_{ij})_{i,j=1}^n$ as follows

$$\begin{gathered} \sum_{i=1}^n (x_i y_0 - y_i x_0)^2 - d^2 {x_0}^2 {y_0}^2 = \sum_{i=1}^n \left[(x_i y_0 + y_i x_0) - 2 y_i x_0\right]^2 - d^2 {x_0}^2 {y_0}^2 = \\ \sum_{i=1}^n \left[z_{i0}^2 - 4\left(x_i y_0 + y_i x_0\right) y_i x_0 + 4 y_i^2 x_0^2\right] - d^2 {x_0}^2 {y_0}^2 = \\ \sum_{i=1}^n \left(z_{i0}^2 - 4 x_i y_0 y_i x_0\right) - d^2 {x_0}^2 {y_0}^2 = \sum_{i=1}^n \left(z_{i0}^2 - 4 \frac{z_{00}}{2} \frac{z_{ii}}{2}\right) - d^2 \frac{z_{00}}{2}^2 \end{gathered} \tag{6.5}$$

arriving at the expression

$$\sum_{i=1}^{n} \left( z_{i0}^2 - z_{00} z_{ii} \right) - \frac{d^2}{4} z_{00}^2 \tag{6.6}$$

Thus, $\mathcal{V}_d$ can be regarded as a quadric in $\mathbb{P}^N$ and it is known as the *quadric of segments* (or QOS for short).

### 6.3.1 Rank of the quadric of segments.

Writing equation 6.6 as

$$\sum_{i=1}^{n} z_{i0}^2 - \left( \sum_{i=1}^{n} z_{ii} \right) z_{00} - \frac{d^2}{4} z_{00}^2$$

and changing variables according to the rule

$$z'_{ij} = \begin{cases} z_{ij} & \text{if} \quad (i,j) \neq (1,1) \\ \frac{id}{2} z_{00} & \text{if} \quad (i,j) = (0,0) \\ -\frac{2}{id} \sum_{i=1}^{n} z_{ii} & \text{if} \quad (i,j) = (1,1) \end{cases}$$

we obtain the following expression for the quadric of segments

$$z'^2_{00} + z'^2_{10} + \cdots + z'^2_{n0} + z'_{11} z'_{00} = 0$$

from which it is obvious that the rank of $\mathcal{V}_d$ is $n+2$.

### 6.3.2 Matrix form.

The matrix representation of the quadratic form given by equation 6.6 is

$$\sigma(\mathbf{x},\mathbf{y})^{\mathtt{T}}\ \mathtt{C}\ \sigma(\mathbf{x},\mathbf{y}) = \sigma(\mathbf{x},\mathbf{y})^{\mathtt{T}} \left( \mathtt{C}_1^{\text{euc}} + \frac{d^2}{2} \mathtt{C}_2^{\text{euc}} \right) \sigma(\mathbf{x},\mathbf{y})$$

where

$$\mathtt{C}_1^{\text{euc}} = \begin{pmatrix} \mathtt{A} & & \\ & \mathtt{I}_n & \\ & & \mathbf{0} \end{pmatrix}, \mathtt{A} = \begin{pmatrix} 0 & -1 & \cdots & -1 \\ -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & \cdots & 0 \end{pmatrix}, \mathtt{C}_2^{\text{euc}} = \begin{pmatrix} -\frac{1}{2} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \tag{6.7}$$

Note that the ranks of $\mathtt{C}_1^{\text{euc}}$ and $\mathtt{C}_2^{\text{euc}}$ are, respectively, $n+2$ and 1.

### 6.3.3 Changes of coordinates.

An homography $\mathtt{H}$ in $\mathbb{P}^n$ such that $\mathbf{x}' = \mathtt{H}\mathbf{x}$ induces an homography $\widetilde{\mathtt{H}}$ in $\mathbb{P}^N$ defined by

$$\sigma(\mathbf{x}', \mathbf{y}') = \widetilde{\mathtt{H}}\, \sigma(\mathbf{x}, \mathbf{y}) \tag{6.8}$$

In order to see how both homographies are related, first note that the $N+1$ vectors $\{\sigma(\mathbf{e}_i, \mathbf{e}_j) \in \mathbb{C}^{N+1} \,|\, 0 \leq i \leq j \leq n\}$ are such that their only non-zero entry is either $\sqrt{2}$ or 1 and it is located at a coordinate univocally determined by $(i, j)$. We can obtain each column $\widetilde{\mathbf{h}}_k$ of $\widetilde{\mathtt{H}}$ in terms of columns $\mathbf{h}_i$ of $\mathtt{H}$:

$$\begin{aligned}
\widetilde{\mathbf{h}}_0 &= \widetilde{\mathtt{H}} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \widetilde{\mathtt{H}} \sigma(\mathbf{e}_0, \mathbf{e}_0) = \frac{1}{\sqrt{2}} \sigma(\mathtt{H}\mathbf{e}_0, \mathtt{H}\mathbf{e}_0) = \frac{1}{\sqrt{2}} \sigma(\mathbf{h}_0, \mathbf{h}_0) \\
&\vdots \\
\widetilde{\mathbf{h}}_{n+1} &= \widetilde{\mathtt{H}} \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} = \widetilde{\mathtt{H}} \sigma(\mathbf{e}_0, \mathbf{e}_1) = \sigma(\mathtt{H}\mathbf{e}_0, \mathtt{H}\mathbf{e}_1) = \sigma(\mathbf{h}_0, \mathbf{h}_1) \\
&\vdots
\end{aligned}$$

So we see that the homography $\widetilde{\mathtt{H}}$ has the following expression as a column matrix

$$\left( \frac{\sigma(\mathbf{h}_0, \mathbf{h}_0)}{\sqrt{2}}, \ldots, \frac{\sigma(\mathbf{h}_n, \mathbf{h}_n)}{\sqrt{2}}, \sigma(\mathbf{h}_0, \mathbf{h}_1), \sigma(\mathbf{h}_0, \mathbf{h}_2), \ldots, \sigma(\mathbf{h}_0, \mathbf{h}_n), \ldots, \sigma(\mathbf{h}_{n-1}, \mathbf{h}_n) \right) \tag{6.9}$$

Let us consider the change of coordinates from an arbitrary reference frame to a Euclidean one given by $\mathbf{s}' = \widetilde{\mathtt{H}}\mathbf{s}$, then

$$\mathbf{s}^{\mathtt{T}} \mathtt{C} \mathbf{s} = \left(\widetilde{\mathtt{H}}^{-1}\mathbf{s}\right)^{\mathtt{T}} \left(\widetilde{\mathtt{H}}^{\mathtt{T}} \mathtt{C} \widetilde{\mathtt{H}}\right) \left(\widetilde{\mathtt{H}}^{-1}\mathbf{s}\right) = \mathbf{s'}^{\mathtt{T}} \mathtt{C}^{\mathrm{euc}} \mathbf{s'}$$

and the quadric of segments is transformed by $\widetilde{\mathtt{H}}$ as

$$\mathtt{C}^{\mathrm{euc}} = \widetilde{\mathtt{H}}^{\mathtt{T}} \mathtt{C} \widetilde{\mathtt{H}} \tag{6.10}$$

$$\mathtt{C_i}^{\mathrm{euc}} = \widetilde{\mathtt{H}}^{\mathtt{T}} \mathtt{C_i} \widetilde{\mathtt{H}}, i \in \{1, 2\} \tag{6.11}$$

**Lemma 6.3.1.** If $\widetilde{\mathtt{H}}$ is an homography in $\mathbb{P}^N$ as defined in 6.8 then $\widetilde{\mathtt{H}^{\mathtt{T}}} = \widetilde{\mathtt{H}}^{\mathtt{T}}$

*Proof.* Using the identity

$$\langle \mathtt{S}(\mathbf{u},\mathbf{v}),\mathtt{S}(\mathbf{w},\mathbf{z})\rangle = \langle \mathbf{u},\mathbf{w}\rangle \langle \mathbf{v},\mathbf{z}\rangle + \langle \mathbf{u},\mathbf{z}\rangle \langle \mathbf{v},\mathbf{w}\rangle$$

noting that $\sigma = \nu \circ \mathtt{S}$, and taking into account the isometry of $\nu$, we get

$$\begin{aligned}
\left\langle \sigma(\mathbf{u},\mathbf{v}), \widetilde{\mathtt{H}}^{\mathtt{T}}\sigma(\mathbf{w},\mathbf{z})\right\rangle &= \left\langle \widetilde{\mathtt{H}}\sigma(\mathbf{u},\mathbf{v}),\sigma(\mathbf{w},\mathbf{z})\right\rangle \\
&= \langle \sigma(\mathtt{H}\mathbf{u},\mathtt{H}\mathbf{v}),\sigma(\mathbf{w},\mathbf{z})\rangle \\
&= \langle \mathtt{H}\mathbf{u},\mathbf{w}\rangle \langle \mathtt{H}\mathbf{v},\mathbf{z}\rangle + \langle \mathtt{H}\mathbf{u},\mathbf{z}\rangle \langle \mathtt{H}\mathbf{v},\mathbf{w}\rangle \\
&= \left\langle \mathbf{u},\mathtt{H}^{\mathtt{T}}\mathbf{w}\right\rangle \left\langle \mathbf{v},\mathtt{H}^{\mathtt{T}}\mathbf{z}\right\rangle + \left\langle \mathbf{u},\mathtt{H}^{\mathtt{T}}\mathbf{z}\right\rangle \left\langle \mathbf{v},\mathtt{H}^{\mathtt{T}}\mathbf{w}\right\rangle \\
&= \left\langle \mathtt{S}(\mathbf{u},\mathbf{v}),\mathtt{S}(\mathtt{H}^{\mathtt{T}}\mathbf{w},\mathtt{H}^{\mathtt{T}}\mathbf{z})\right\rangle \\
&= \left\langle \sigma(\mathbf{u},\mathbf{v}),\sigma(\mathtt{H}^{\mathtt{T}}\mathbf{w},\mathtt{H}^{\mathtt{T}}\mathbf{z})\right\rangle \\
&= \left\langle \sigma(\mathbf{u},\mathbf{v}),\widetilde{\mathtt{H}^{\mathtt{T}}}\sigma(\mathbf{w},\mathbf{z})\right\rangle
\end{aligned}$$

The last equality holds for every point in $\mathbb{P}^N$ because $\sigma$ is surjective. □

**Theorem 6.3.1.** Let the hyperplane at infinity be $\boldsymbol{\pi}_\infty \in \mathbb{C}^{n+1}$ then

$$\mathtt{C}_2 \sim \sigma(\boldsymbol{\pi}_\infty,\boldsymbol{\pi}_\infty)\sigma(\boldsymbol{\pi}_\infty,\boldsymbol{\pi}_\infty)^{\mathtt{T}}$$

*Proof.* Let $\mathtt{H}$ be an homography having $\mathbf{r}_i$ as row vectors with $i \in \{0,\ldots,n\}$. By the previous lemma and the expression for $\widetilde{\mathtt{H}}$ in terms of $\mathtt{H}$, we have that

$$\widetilde{\mathtt{H}} = \widetilde{\mathtt{H}^{\mathtt{T}}}^{\mathtt{T}} = \left(\frac{\sigma(\mathbf{r}_0,\mathbf{r}_0)}{\sqrt{2}},\ldots,\frac{\sigma(\mathbf{r}_n,\mathbf{r}_n)}{\sqrt{2}},\sigma(\mathbf{r}_0,\mathbf{r}_1),\sigma(\mathbf{r}_0,\mathbf{r}_2),\ldots,\sigma(\mathbf{r}_0,\mathbf{r}_n),\ldots,\sigma(\mathbf{r}_{n-1},\mathbf{r}_n)\right) \tag{6.12}$$

Taking into account 6.7,

$$\mathtt{C}_2^{\mathrm{euc}}\widetilde{\mathtt{H}} \sim \begin{pmatrix} \sigma(\mathbf{r}_0,\mathbf{r}_0)^{\mathtt{T}} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and substituting this into

$$\mathtt{C}_2 \sim \widetilde{\mathtt{H}}^{\mathtt{T}}\mathtt{C}_2^{\mathrm{euc}}\widetilde{\mathtt{H}} \sim \widetilde{\mathtt{H}}^{\mathtt{T}}\left(\mathtt{C}_2^{\mathrm{euc}}\widetilde{\mathtt{H}}\right)$$

we get

$$\mathtt{C}_2 \sim \sigma(\mathbf{r}_0,\mathbf{r}_0)\sigma(\mathbf{r}_0,\mathbf{r}_0)^{\mathtt{T}}$$

The result follows fixing an homography $\mathtt{H}$ such that $\mathbf{r}_0 \sim \boldsymbol{\pi}_\infty$. □

### 6.3.4 Linear obtainment of the quadric of segments.

Our aim is to use the quadric of segments to obtain a Euclidean upgrading of a projective reconstruction of a scene. As a first step, we need to locate it in the projective calibration which means we will have to find the quadric of segments in an arbitrary coordinate system.

Each non-oriented segment $\{\mathbf{x}, \mathbf{y}\}$ for a given $d$ imposes a linear constraint $\sigma(\mathbf{x}, \mathbf{y})^{\mathtt{T}}\ \mathtt{C}\ \sigma(\mathbf{x}, \mathbf{y}) = 0$ on the entries of $\mathtt{C}$. The segments don't have to be of the same length because we can write

$$\sigma(\mathbf{x}, \mathbf{y})^{\mathtt{T}}\ \mathtt{C}\ \sigma(\mathbf{x}, \mathbf{y}) = \sigma(\mathbf{x}, \mathbf{y})^{\mathtt{T}} \left( \mathtt{C}_1 + \frac{d_i^2}{2} \mathtt{C}_2 \right) \sigma(\mathbf{x}, \mathbf{y}) = 0 \tag{6.13}$$

We are going to see that $L = \dim \mathrm{Sym}(N+1) - 1$ of such constraints determine the quadric of segments uniquely (up to a scale factor).

As a result of Theorem 6.3.1, we can parameterize the space $\mathtt{S}_2$ of every possible $\mathtt{C}_2$ matrix by the mapping

$$\begin{aligned} \mathtt{C}_2 : \mathbb{P}^n &\longrightarrow \mathbb{P}^{N \times N} \\ \mathbf{x} &\mapsto \mathtt{C}_2(\mathbf{x}) = \sigma(\mathbf{x}, \mathbf{x}) \sigma(\mathbf{x}, \mathbf{x})^{\mathtt{T}} \end{aligned}$$

The entries of $\sigma(\mathbf{x}, \mathbf{x})$ span every homogeneous monomial of total degree 2 in $\mathbb{C}[x_0, \ldots, x_n]$ and, as a consequence of this, the entries of $\sigma(\mathbf{x}, \mathbf{x})\sigma(\mathbf{x}, \mathbf{x})^{\mathtt{T}}$ span every homogeneous monomial of total degree 4 in $\mathbb{C}[x_0, \ldots, x_n]$. Hence, the dimension of the linear space spanned by the $\mathtt{C}_2$ matrices is

$$\dim \mathtt{S}_2 = \binom{n+4}{4}$$

Now consider the equation

$$\sigma(\mathbf{x}, \mathbf{x})^{\mathtt{T}} \left( \mathtt{C}_1 + 0\mathtt{C}_2 \right) \sigma(\mathbf{x}, \mathbf{x}) = \sigma(\mathbf{x}, \mathbf{x})^{\mathtt{T}} \mathtt{C}_1 \sigma(\mathbf{x}, \mathbf{x}) = 0$$

which holds for every $\mathbf{x}$ because the distance of a point to itself is always zero. From this and the properties of the trace,

$$\begin{aligned} 0 = \mathrm{trace}(0) &= \mathrm{trace}\left(\sigma(\mathbf{x}, \mathbf{x})^{\mathtt{T}} \mathtt{C}_1 \sigma(\mathbf{x}, \mathbf{x})\right) \\ &= \mathrm{trace}\left(\mathtt{C}_1 \sigma(\mathbf{x}, \mathbf{x}) \sigma(\mathbf{x}, \mathbf{x})^{\mathtt{T}}\right) \\ &= \langle \mathtt{C}_1, \mathtt{C}_2(\mathbf{x}) \rangle \end{aligned}$$

Consequently, the subspace $\mathtt{S}_1$ of every possible $\mathtt{C}_1$ matrix is orthogonal (remember that we are using $\frac{1}{2}\mathrm{trace}(\mathtt{AB})$ as the scalar product of two matrices) to the space $\mathtt{S}_2$ and this leads us to an upper bound for the dimension of $\mathtt{S}_1$

$$\dim \mathtt{S}_1 \leq \dim \mathrm{Sym}(N+1) - \dim \mathtt{S}_2 \tag{6.14}$$

It can be proved that this is indeed an equality.

We can obtain an orthonormal basis $\{\mathtt{M}^{(2)}_\beta \in \mathbb{C}^{(N+1)\times(N+1)} \mid \beta \in \{1,\dots,\dim \mathtt{S}_2\}\}$ of $\mathtt{S}_2$ and use it to solve the system of $\dim \mathtt{S}_2$ linear equations

$$\left\langle \mathtt{C}_1, \mathtt{M}^{(2)}_\beta \right\rangle = 0.$$

In this way, we obtain a parameterization of the elements of $\mathtt{S}_1$ from which it is easy to compute a basis $\{\mathtt{M}^{(1)}_\alpha \in \mathbb{C}^{(N+1)\times(N+1)} \mid \alpha \in \{1,\dots,\dim \mathtt{S}_1\}\}$. We can now restate the problem of locating the quadric of segments as solving the linear system in $L+1$ unknowns

$$\sigma(\mathbf{x}_i,\mathbf{y}_i)^{\mathtt{T}} \left( \sum_{\alpha=1}^{\dim \mathtt{S}_1} a^{(1)}_\alpha \mathtt{M}^{(1)}_\alpha + \frac{d_i^2}{2} \sum_{\beta=1}^{\dim \mathtt{S}_1} a^{(2)}_\beta \mathtt{M}^{(2)}_\beta \right) \sigma(\mathbf{x}_i,\mathbf{y}_i) = 0.$$

Thus, the quadric of segments can be recovered up to a non-zero scale factor by knowing $L$ segments in general position and solving the corresponding system of equations. Note that in the 3D case this amounts to knowing the end points and lengths of 55 segments in the original scene.

## 6.4 Affine and Euclidean upgrading from the QOS.

### 6.4.1 Relationship between $\mathtt{C}_2$ and the hyperplane at infinity.

**Theorem 6.4.1.** Let $\alpha,\beta,\gamma \in \{0,\dots,n\}$. Then

$$\boldsymbol{\pi}_\infty \sim \begin{pmatrix} \sigma(\mathbf{e_0},\mathbf{e}_{\boldsymbol{\alpha}})^{\mathtt{T}}\, \mathtt{C}_2\, \sigma(\mathbf{e}_{\boldsymbol{\beta}},\mathbf{e}_{\boldsymbol{\gamma}}) \\ \vdots \\ \sigma(\mathbf{e_n},\mathbf{e}_{\boldsymbol{\alpha}})^{\mathtt{T}}\, \mathtt{C}_2\, \sigma(\mathbf{e}_{\boldsymbol{\beta}},\mathbf{e}_{\boldsymbol{\gamma}}) \end{pmatrix}.$$

*Proof.* From the proof of 6.3.1, we know that $\mathtt{C}_2(\mathbf{x}) = \sigma(\mathbf{x},\mathbf{x})\sigma(\mathbf{x},\mathbf{x})^{\mathtt{T}}$ and we can write for each $0 \le i \le n$

$$\sigma(\mathbf{e_i},\mathbf{e}_{\boldsymbol{\alpha}})^{\mathtt{T}}\, \mathtt{C}_2\, \sigma(\mathbf{e}_{\boldsymbol{\beta}},\mathbf{e}_{\boldsymbol{\gamma}}) = \sigma(\mathbf{e_i},\mathbf{e}_{\boldsymbol{\alpha}})^{\mathtt{T}}\sigma(\mathbf{x},\mathbf{x})\sigma(\mathbf{x},\mathbf{x})^{\mathtt{T}}\sigma(\mathbf{e}_{\boldsymbol{\beta}},\mathbf{e}_{\boldsymbol{\gamma}}).$$

Observe that the first two factors in the right hand side of the previous equation can be expressed as

$$\begin{aligned}\sigma(\mathbf{e_i}, \mathbf{e_\alpha})^{\mathtt{T}} \sigma(\mathbf{x}, \mathbf{x}) &= \nu(\mathtt{S}(\mathbf{e_i}, \mathbf{e_\alpha}))^{\mathtt{T}} \nu(\mathtt{S}(\mathbf{x}, \mathbf{x})) \\ &= \frac{1}{2} \operatorname{trace}(\mathtt{S}(\mathbf{e_i}, \mathbf{e_\alpha}) \mathtt{S}(\mathbf{x}, \mathbf{x})) \\ &= \frac{1}{2} \operatorname{trace}((\mathbf{e_i} \mathbf{e_\alpha}^{\mathtt{T}} + \mathbf{e_\alpha} \mathbf{e_i}^{\mathtt{T}})(2\mathbf{x}\mathbf{x}^{\mathtt{T}})) \\ &= 2x_i x_\alpha.\end{aligned}$$

Thus, $\sigma(\mathbf{x}, \mathbf{x})^{\mathtt{T}} \sigma(\mathbf{e_\beta}, \mathbf{e_\gamma}) = 2x_\beta x_\gamma$ and combining both factors we have that

$$\sigma(\mathbf{e_i}, \mathbf{e_\alpha})^{\mathtt{T}}\, \mathtt{C}_2\, \sigma(\mathbf{e_\beta}, \mathbf{e_\gamma}) = 4x_i x_\alpha x_\beta x_\gamma.$$

Particularizing for $\mathbf{x} = \boldsymbol{\pi}_\infty$ we get the desired result. □

### 6.4.2 Relationship between $\mathtt{C}_1$ and the AQC.

Recall from the previous chapter that the entries of the Plücker matrix associated to a line determined by two vectors $\mathbf{x}$ and $\mathbf{y}$ are given by the relation $m_{ij} = x_i y_j - x_j y_i$. Consequently,

$$m_{ij} m_{kl} = z_{il} z_{jk} - z_{ik} z_{jl}$$

where $z_{ij} = x_i y_j + x_j y_i$. Thus, we can express any quadric in Plücker coordinates as a quadric in symmetric coordinates. In particular, one may rewrite the equation for the absolute quadratic in this fashion, as can be seen in the following result.

**Theorem 6.4.2.** Let $\mathbf{x}, \mathbf{y} \in \mathbb{C}^4$. Then

$$(\mathbf{x} \wedge \mathbf{y})^{\mathtt{T}}\, \Sigma\, (\mathbf{x} \wedge \mathbf{y}) = \sigma(\mathbf{x}, \mathbf{y})^{\mathtt{T}}\, \mathtt{C}_1\, \sigma(\mathbf{x}, \mathbf{y}) \qquad (6.15)$$

*Proof.* If $\mathbf{x}$ and $\mathbf{y}$ are in Euclidean coordinates then

$$\begin{aligned}\sigma(\mathbf{x}, \mathbf{y})^{\mathtt{T}}\, \mathtt{C_1}^{\text{euc}}\, \sigma(\mathbf{x}, \mathbf{y}) &= \sum_{i=1}^{n} (x_i y_0 - y_i x_0)^2 = \sum_{i=1}^{n} m_{i0}^2 \\ &= (\mathbf{x} \wedge \mathbf{y})^{\mathtt{T}}\, \Sigma^{\text{euc}}\, (\mathbf{x} \wedge \mathbf{y})\end{aligned}$$

and since changes of coordinates do not affect the identity 6.15, this establishes the result. □

Both sides of equation 6.15 are polynomials of total degree four in $\mathbb{C}[x_0, \ldots, x_n, y_0, \ldots, y_n]$, so taking fourth order derivatives we can write the entries of $\mathtt{C_1}$ in terms of those of $\Sigma$. In particular, in the 3D case with $\Sigma = (s_{ij})_{i,j=1}^{6} \in \mathbb{C}^4$ the matrix representing $\mathtt{C_1}$ is

$$\begin{pmatrix}
0 & -s_{11} & -s_{22} & -s_{33} & 0 & 0 & 0 & -s_{12}\sqrt{2} & -s_{13}\sqrt{2} & -s_{23}\sqrt{2} \\
-s_{11} & 0 & -s_{44} & -s_{55} & 0 & s_{14}\sqrt{2} & s_{15}\sqrt{2} & 0 & 0 & -s_{45}\sqrt{2} \\
-s_{22} & -s_{44} & 0 & -s_{66} & -s_{24}\sqrt{2} & 0 & s_{26}\sqrt{2} & 0 & s_{46}\sqrt{2} & 0 \\
-s_{33} & -s_{55} & -s_{66} & 0 & -s_{35}\sqrt{2} & -s_{36}\sqrt{2} & 0 & -s_{56}\sqrt{2} & 0 & 0 \\
0 & 0 & -s_{24}\sqrt{2} & -s_{35}\sqrt{2} & s_{11} & s_{12} & s_{13} & -s_{14} & -s_{15} & -s_{25} - s_{34} \\
0 & s_{14}\sqrt{2} & 0 & -s_{36}\sqrt{2} & s_{12} & s_{22} & s_{23} & s_{24} & s_{34} - s_{16} & -s_{26} \\
0 & s_{15}\sqrt{2} & s_{26}\sqrt{2} & 0 & s_{13} & s_{23} & s_{33} & s_{25} + s_{16} & s_{35} & s_{36} \\
-s_{12}\sqrt{2} & 0 & 0 & -s_{56}\sqrt{2} & -s_{14} & s_{24} & s_{25} + s_{16} & s_{44} & s_{45} & -s_{46} \\
-s_{13}\sqrt{2} & 0 & s_{46}\sqrt{2} & 0 & -s_{15} & s_{34} - s_{16} & s_{35} & s_{45} & s_{55} & s_{56} \\
-s_{23}\sqrt{2} & -s_{45}\sqrt{2} & 0 & 0 & -s_{25} - s_{34} & -s_{26} & s_{36} & -s_{46} & s_{56} & s_{66}
\end{pmatrix}$$

and this together with the equation arising from the constraint $\mathrm{trace}(\Omega\,\Sigma) = 0$ allows us to write the entries of $\Sigma$ in terms of those of $\mathtt{C_1}$.

As a consequence of the previous discussion, the knowledge of the quadric of segments in an arbitrary reference frame allows us to recover the absolute quadratic complex. At this point, we can use the techniques developed in the previous chapter to attain our objective of computing a rectifying homography.

# Chapter 7

# Conclusion

We have conducted an in-depth study of several geometric entities that allow us to perform a Euclidean upgrading of a projective reconstruction. In doing so, we have not discussed some practical matters like how to account for the presence of noise, etc. and have focused instead in the more geometrical side of the subject.

# Bibliography


[GvL96] G. H. Golub and C. F. van Loan. *Matrix Computations*. Johns Hopkins University Press, 3rd edition, 1996.

[HZ04] R. I. Hartley and A. Zisserman. *Multiple View Geometry in Computer Vision*. Cambridge University Press, ISBN: 0521540518, second edition, 2004.

[PMP+05] J. Ponce, K. McHenry, T. Papadopoulo, M. Teillaud, and B. Triggs. On the absolute quadratic complex and its application to autocalibration. In *CVPR '05: Proceedings of the 2005 IEEE Computer Society Conference on Computer Vision and Pattern Recognition (CVPR'05) - Volume 1*, pages 780–787, Washington, DC, USA, 2005. IEEE Computer Society.

[PW01] Helmut Pottmann and Johannes Wallner. *Computational Line Geometry*. Springer-Verlag New York, Inc., Secaucus, NJ, USA, 2001.

[RV10] José I. Ronda and Antonio Valdés. Euclidean upgrading from segment lengths. *Preprint*, 2010.

[RVG08] José I. Ronda, Antonio Valdés, and Guillermo Gallego. Line geometry and camera autocalibration. *J. Math. Imaging Vis.*, 32(2):193–214, 2008.

[SK79] J. G. Semple and G. T. Kneebone. *Algebraic Projective Geometry*. 1979.

[Tri97] Bill Triggs. Autocalibration and the absolute quadric. In *CVPR*, pages 609–614, 1997.

[Wik09] Wikipedia. Charge-coupled device — wikipedia, the free encyclopedia, 2009. [Online; accessed 6-October-2009].